17 April 2013 **Name:** Solution

ECE 438 **Exam No. 3** **Spring 2013**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider two random variables X and Y which are jointly distributed according to the following bivariate density function

$$f_{XY}(x,y)=\begin{cases}2, & 0\le y\le 1-x,\, 0\le x\le 1\\ 0, & \text{else}\end{cases}$$

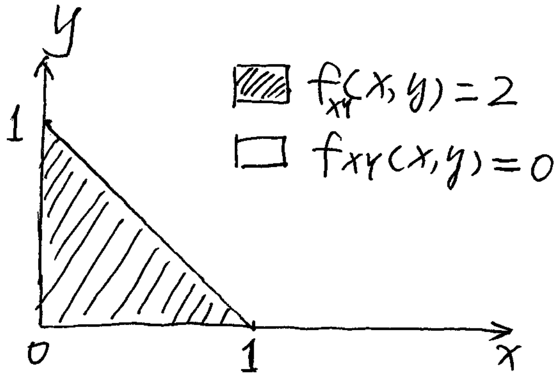
(3) a. Sketch $f_{XY}(x,y)$.

(6) b. Find the marginal densities $f_X(x)$ and $f_Y(y)$.

(1) c. Are X and Y independent?

(15) d. Find the mean and variance of X and Y and the correlation coefficient ρ_{XY} between them.

a)



b)

$$f_X(x)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$$

$$=\int_0^{1-x} 2\,dy$$

$$=2(1-x) \qquad 0\le x\le 1$$

$$f_Y(y)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$$

$$=\int_0^{1-y} 2\,dx$$

$$=2(1-y), \qquad 0\le y\le 1.$$

$$\therefore \quad f_X(x) = \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

$$f_Y(y) = \begin{cases} 2(1-y), & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

c) No.

$$\because \quad f_{XY}(x,y) \neq f_X(x) f_Y(y)$$

d) Mean:

$$\begin{aligned} E\{X\} &= \int_{-\infty}^{+\infty} x f_X(x) dx \\ &= \int_0^1 2x(1-x) dx \\ &= 2\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1 \\ &= 2\left(\tfrac{1}{2} - \tfrac{1}{3}\right) \\ &= \tfrac{1}{3} \end{aligned}$$

Similarly, $E\{Y\} = \frac{1}{3}$.

Variance:

$$\begin{aligned} E\{X^2\} &= \int_{-\infty}^{+\infty} x^2 f_X(x) dx \\ &= \int_0^1 2x^2(1-x) dx \\ &= 2\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^1 \\ &= 2\left(\tfrac{1}{3} - \tfrac{1}{4}\right) \\ &= \tfrac{1}{6} \end{aligned}$$

$$\therefore \quad \sigma_x^2 = E\{X^2\} - E^2\{X\}$$
$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2$$
$$= \frac{1}{18}$$

Similarly, $\sigma_Y^2 = \frac{1}{18}$

ρ_{XY}:

$$E\{XY\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{XY}(x,y)\,dx\,dy$$
$$= \int_0^1 x\,dx \int_0^{1-x} 2y\,dy$$
$$= \int_0^1 x(1-x)^2\,dx$$
$$= \int_0^1 (x - 2x^2 + x^3)\,dx$$
$$= \left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^1$$
$$= \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$
$$= \frac{1}{12}$$

$$\therefore \quad \rho_{XY} = \frac{E\{XY\} - E\{X\}E\{Y\}}{\sigma_X \sigma_Y}$$
$$= \frac{\frac{1}{12} - \frac{1}{3} \times \frac{1}{3}}{\frac{1}{18}}$$
$$= -\frac{1}{2}$$

2. (25) Consider a random process comprised of an independent, identically distributed sequence of random variables $X_n, -\infty < n < \infty$ with mean μ_X and variance σ_X^2. Suppose we define two new random processes Y_n and Z_n according to the equations

$$Y_n = \frac{1}{2}\left(X_n + X_{n-1}\right)$$

$$Z_n = \frac{1}{2}\left(X_n - X_{n-1}\right)$$

a. (15) Find the means $\mu_Y[n] = E\left\{Y_n\right\}, \mu_Z[n] = E\left\{Z_n\right\}$, and autocorrelation functions $r_{YY}[m,n] = E\left\{Y_m Y_n\right\}, r_{ZZ}[m,n] = E\left\{Z_m Z_n\right\}$ for these two random processes.

b. (10) Find the cross-correlation function $r_{YZ}[m,n] = E\left\{Y_m Z_n\right\}$.

Note: The solution for part a. below is correct; but it is a brute-force approach. A much simpler and more efficient way to solve the problem would be to use the formulas from the formula sheet shown to the right.

$$\mu_Y = \mu_X \sum_m h[m]$$

$$r_{XY}[n] = h[n] * r_{XX}[n]$$

$$r_{YY}[n] = h[n] * r_{XY}[-n]$$

2. a. $E[Y_n] = E[\frac{1}{2}(X_n + X_{n-1})] = \frac{1}{2}E[X_n] + \frac{1}{2}E[X_{n-1}]$

$= \mu_X$

$E[Z_n] = E[\frac{1}{2}(X_n - X_{n-1})] = \frac{1}{2}E[X_n] - \frac{1}{2}E[X_{n-1}] = 0$

$r_{YY}(m,n) = E[Y_m Y_n]$

$= E[(\frac{1}{2}(X_m + X_{m-1}))\cdot\frac{1}{2}(X_n + X_{n-1})]$

$= \frac{1}{4}E[X_m X_n + X_m X_{n-1} + X_{m-1}X_n + X_{m-1}X_{n-1}]$

If $m=n$, $E[X_m X_n] = E[X_{m-1}X_{n-1}] = \sigma_X^2 + \mu_X^2$

$E[X_m X_{n-1}] = E[X_{m-1}X_n] = E[X_m]E[X_{n-1}] = \mu_X^2$

$\Rightarrow r_{YY}(m,n) = \frac{1}{4}(2\sigma_X^2 + 2\mu_X^2 + 2\mu_X^2) = \frac{1}{2}\sigma_X^2 + \mu_X^2$

If $m=n-1$, $E[X_m X_n] = E[X_{m-1}X_{n-1}] = \mu_X^2 = E[X_{m-1}X_n]$

$E[X_m X_{n-1}] = \sigma_X^2 + \mu_X^2$

$\Rightarrow r_{YY}(m,n) = \frac{1}{4}(3\mu_X^2 + \sigma_X^2 + \mu_X^2) = \frac{1}{4}\sigma_X^2 + \mu_X^2$

If $m=n+1$, $E[X_m X_n] = E[X_{m-1}X_{n-1}] = E[X_m X_{n-1}] = \mu_X^2$

$E[X_{m-1}X_n] = \sigma_X^2 + \mu_X^2$

$\Rightarrow r_{YY}(m,n) = \frac{1}{4}(3\mu_X^2 + \sigma_X^2 + \mu_X^2) = \frac{1}{4}\sigma_X^2 + \mu_X^2$

else $E[X_m X_n] = E[X_m X_{n-1}] = E[X_{m-1}X_n] = E[X_{m-1}X_{n-1}] = \mu_X^2$

~~In general~~ $r_{YY}(m,n) = \Big\{ \ \frac{1}{4}(\mu_X^2 + \mu_X^2 + \mu_X^2 + \mu_X^2) = \mu_X^2$

$$r_{zz}(m,n) = E[Z_m Z_n] = E[\tfrac{1}{2}(X_m - X_{m-1})\,\tfrac{1}{2}(X_n - X_{n-1})]$$

$$= \tfrac{1}{4} E[X_m X_n - X_m X_{n-1} - X_{m-1} X_n + X_{m-1} X_{n-1}]$$

Using a similar argument

If $m=n$ $\quad r_{zz}(m,n) = \frac{1}{4}(2\sigma_x^2 + 2\mu_x^2 - 2\mu_x^2) = \frac{1}{2}\sigma_x^2$

If $m=n-1$ $\quad r_{zz}(m,n) = \frac{1}{4}(2\mu_x^2 - \mu_x^2 - \sigma_x^2 - \mu_x^2) = -\frac{1}{4}\sigma_x^2$

If $m=n+1$ $\quad r_{zz}(m,n) = \frac{1}{4}(2\mu_x^2 - \mu_x^2 - \sigma_x^2 - \mu_x^2) = -\frac{1}{4}\sigma_x^2$

else $\quad r_{zz}(m,n) = \frac{1}{4}(\mu_x^2 - \mu_x^2 - \mu_x^2 + \mu_x^2) = 0$

(b) $r_{yz}(m,n) = E[Y_m Z_n] = E[\tfrac{1}{2}(X_m + X_{m-1}) \cdot \tfrac{1}{2}(X_n - X_{n-1})]$

$$= \tfrac{1}{4} E[X_m X_n - X_{m-1} X_{n-1} + X_{m-1} X_n - X_m X_{n-1}]$$

Using the same argument as in part (a)

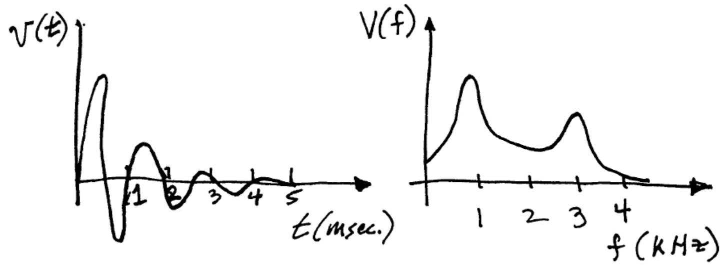
If $m=n$ $\quad r_{yz}(m,n) = 0$

If $m=n-1$ $\quad r_{yz}(m,n) = \frac{1}{4}(\mu_x^2 - \sigma_x^2 - \mu_x^2) = -\frac{1}{4}\sigma_x^2$

If $m=n+1$ $\quad r_{yz}(m,n) = \frac{1}{4}(\sigma_x^2 + \mu_x^2 - \mu_x^2) = \frac{1}{4}\sigma_x^2$

else $\quad r_{yz}(m,n) = \frac{1}{4}(\mu_x^2 - \mu_x^2 + \mu_x^2 - \mu_x^2) = 0$

3. (25) Consider a voiced phoneme for which the time-domain continuous-time vocal tract response $v(t)$ and corresponding frequency response (CTFT) $V(f)$ are given below.

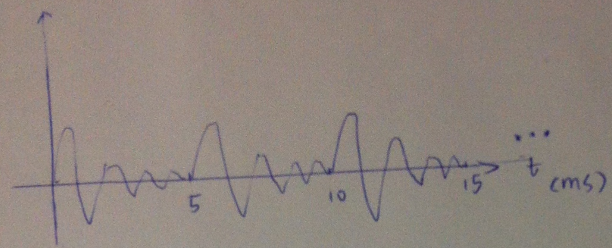


a. (7) Assume that the pitch frequency for the speaker is 200 Hz. Sketch what the continuous-time domain speech waveform $s(t)$ would look like in this case. Be sure to dimension all important quantities in the speech waveform.

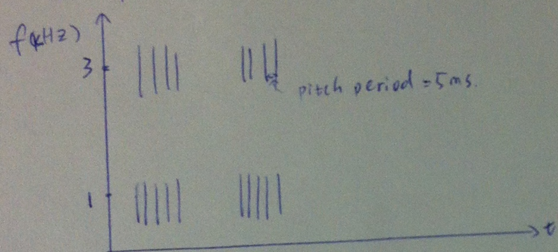
b. (9) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 25 samples. Carefully sketch the resulting spectrogram. Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

c. (9) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 1,000 samples. Carefully sketch the resulting spectrogram. Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

3. (a) pitch period = $\frac{1}{200Hz}$ = 5ms

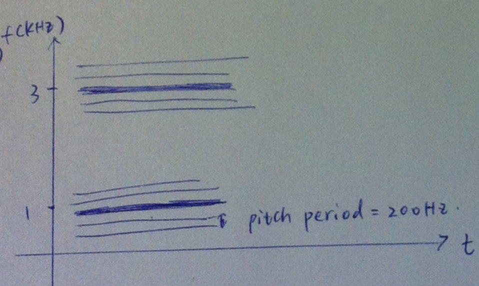


(b) According to the given V(f),
the formant frequencies are $F_1 = 1$ KHz, $F_2 = 3$ KHz.



This is a wideband spectrogram because we have a shorter window length.
which gives good time domain resolution but poor frequency domain resolution.

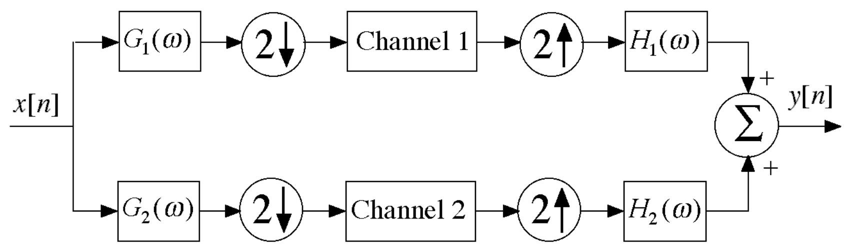
(c)



This is a narrow-band spectrogram. because we have a long-time window length.
That gives good freqnency domain resolution but poor time domain resolution.

4. (25) In class, we derived conditions for perfect reconstruction using an L-channel modulated filter bank in which the ℓ-th channel has unit sample response $h_\ell[n] = h_0[n]e^{j2\pi\ell n/L}$, where $h_0[n]$ is the unit sample response of the 0-th channel. Thus the frequency response of each channel is just a shifted version of the frequency response of the 0-th channel.

In this problem, we will consider the more general type of 2-channel filter bank shown below:



a. (19) The filters $G_1(\omega)$ and $G_2(\omega)$ are referred to as the analysis filters, since they analyze the signal $x[n]$ into its constituent frequency components (in this case, usually the low-pass and high-pass bands, respectively). The filters $H_1(\omega)$ and $H_2(\omega)$ are called the synthesis filters, since they are used to synthesis the complete signal $y[n]$ from the frequency band components of $x[n]$. Derive conditions on the frequency responses of these four filters that guarantee perfect reconstruction with this system, i.e. $y[n] \equiv x[n]$.

b. (6) Show that two ideal band-pass filters (low-pass and high-pass) with cutoff at $\pi / 2$ radians/sample will satisfy these conditions.

Note that the channels at the center of the system are only shown to indicate where quantization, coding, and transmission, or storage would take place. Here we are only interested in the filter bank; so you should assume that both channels are ideal, and have no effect on the signal.

Question 4

a. Consider the frequency response of the first channel

$$X(w) \to \boxed{G_1(w)} \xrightarrow{U(w)} (2\downarrow) \xrightarrow{V(w)} (2\uparrow) \xrightarrow{W(w)} \boxed{H_1(w)} \to X_1(w)$$

$$U(w) = X(w)\,G_1(w)$$

$$V(w) = \frac{1}{2}\sum_{k=0}^{1} U\left(\frac{w-2\pi k}{2}\right) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{w-2\pi k}{2}\right) G_1\left(\frac{w-2\pi k}{2}\right)$$

$$W(w) = V(2w) = \frac{1}{2}\sum_{k=0}^{1} X(w-\pi k)\,G_1(w-\pi k)$$

$$X_1(w) = W(w)\,H_1(w) = \frac{1}{2}\sum_{k=0}^{1} X(w-\pi k)\,G_1(w-\pi k)\,H_1(w)$$

Similarly, for channel 2 we have

$$X_2(w) = \frac{1}{2}\sum_{k=0}^{1} X(w-\pi k)\,G_2(w-\pi k)\,H_2(w)$$

For perfect reconstruction. i.e. $y[n] = x[n]$

It is equivalent to have $Y(w) = X(w)$

therefore $Y(w) = X_1(w) + X_2(w) = X(w)$

$$\Leftrightarrow \quad X(w) = \frac{1}{2}\sum_{k=0}^{1} X(w-\pi k)\left[G_1(w-k\pi)H_1(w) + G_2(w-k\pi)H_2(w)\right]$$

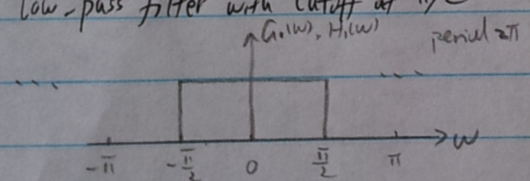
$$\Leftrightarrow \quad X(w) = \frac{1}{2}X(w)\left[G_1(w)H_1(w) + G_2(w)H_2(w)\right] + \frac{1}{2}X(w-\pi)\left[G_1(w-\pi)H_1(w) + G_2(w-\pi)H_2(w)\right]$$

therefore, for perfect reconstruction we need

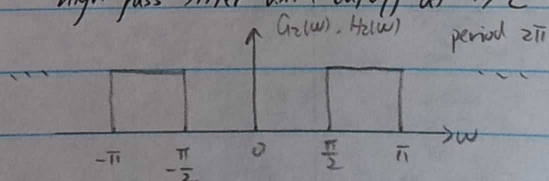
$$G_1(w)H_1(w) + G_2(w)H_2(w) = 2$$

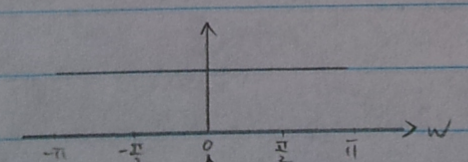
$$G_1(w-\pi)H_1(w) + G_2(w-\pi)H_2(w) = 0$$

b. Low-pass filter with cutoff at $\pi/2$

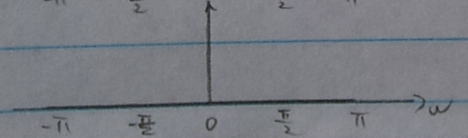


high-pass filter with cutoff at $\pi/2$





$G_1(w)H_1(w) + G_2(w)H_2(w)$ is all pass filter



$G_1(w-\pi)H_1(w) + G_2(w-\pi)H_2(w)$ is all stop filter

thus satisfy perfect reconstruction condition