ECE438 Spring 2013 Exam No. 2

Solution for Problem 1.

a. $\because \; y[n] = x[n] + \frac{1}{2} y[n-1]$

$\xrightarrow{ZT} \; Y(z) = X(z) + \frac{1}{2} z^{-1} Y(z)$

$(1 - \frac{1}{2} z^{-1}) Y(z) = X(z).$

$\Rightarrow \; H(z) = Y(z) / X(z) = \dfrac{1}{1 - \frac{1}{2} z^{-1}}. \qquad |z| > \frac{1}{2}.$

$\because \; x[n] = u[-n+1], \quad n = 1, 0, -1, \ldots$

$$\therefore \; X(z) = \sum_{n=-\infty}^{1} u[-n+1] z^{-n}$$

$$= \sum_{n=-1}^{+\infty} u[n+1] z^{n}$$

$$= z^{-1} + \sum_{n=0}^{+\infty} u[n] z^{n}$$

$$= z^{-1} + \frac{1}{1-z}$$

$$= -\frac{z^{-2}}{1-z^{-1}}, \qquad |z| < 1.$$

$$\therefore \; Y(z) = H(z) X(z) = -\frac{z^{2}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}, \quad \frac{1}{2} < |z| < 1$$

b. $$Y(z) = \frac{-z^{2}}{1/2\, z^{-2} - \frac{3}{2} z^{-1} + 1} = -2 + \frac{-3z^{-1} + 2}{\frac{1}{2} z^{-2} - \frac{3}{2} z^{-1} + 1} = -2 + \frac{-3z^{-1}+2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

Let $Y(z) = -2 + Y_1(z)$. $\; Y_1(z) = \dfrac{-3z^{-1}+2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} = \dfrac{A}{1-\frac{1}{2}z^{-1}} + \dfrac{B}{1-z^{-1}}.$

~~$Y_1(z) =$~~ $\therefore \; A = \left.\dfrac{-3z^{-1}+2}{1-z^{-1}}\right|_{z=\frac{1}{2}} = 4.$

$B = \left.\dfrac{-3z^{-1}+2}{1-\frac{1}{2}z^{-1}}\right|_{z=1} = -2.$

check : $\dfrac{4}{1-\frac{1}{2}z^{-1}} + \dfrac{-2}{1-z^{-1}} = \dfrac{-3z^{-1}+2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}.$

$\because \frac{1}{2} < |z| < 1$, $\quad \delta[n] \xleftrightarrow{ZT} 1$.

$\therefore \quad \dfrac{4}{1-\frac{1}{2}z^{-1}} \xrightarrow{ZT^{-1}} 4\left(\frac{1}{2}\right)^n u[n]$

$\dfrac{-2}{1-z^{-1}} \xrightarrow{ZT^{-1}} +2u[-n-1]$

$\therefore \quad y[n] = -2\delta[n] + 4\left(\frac{1}{2}\right)^n u[n] + 2u[-n-1]$.

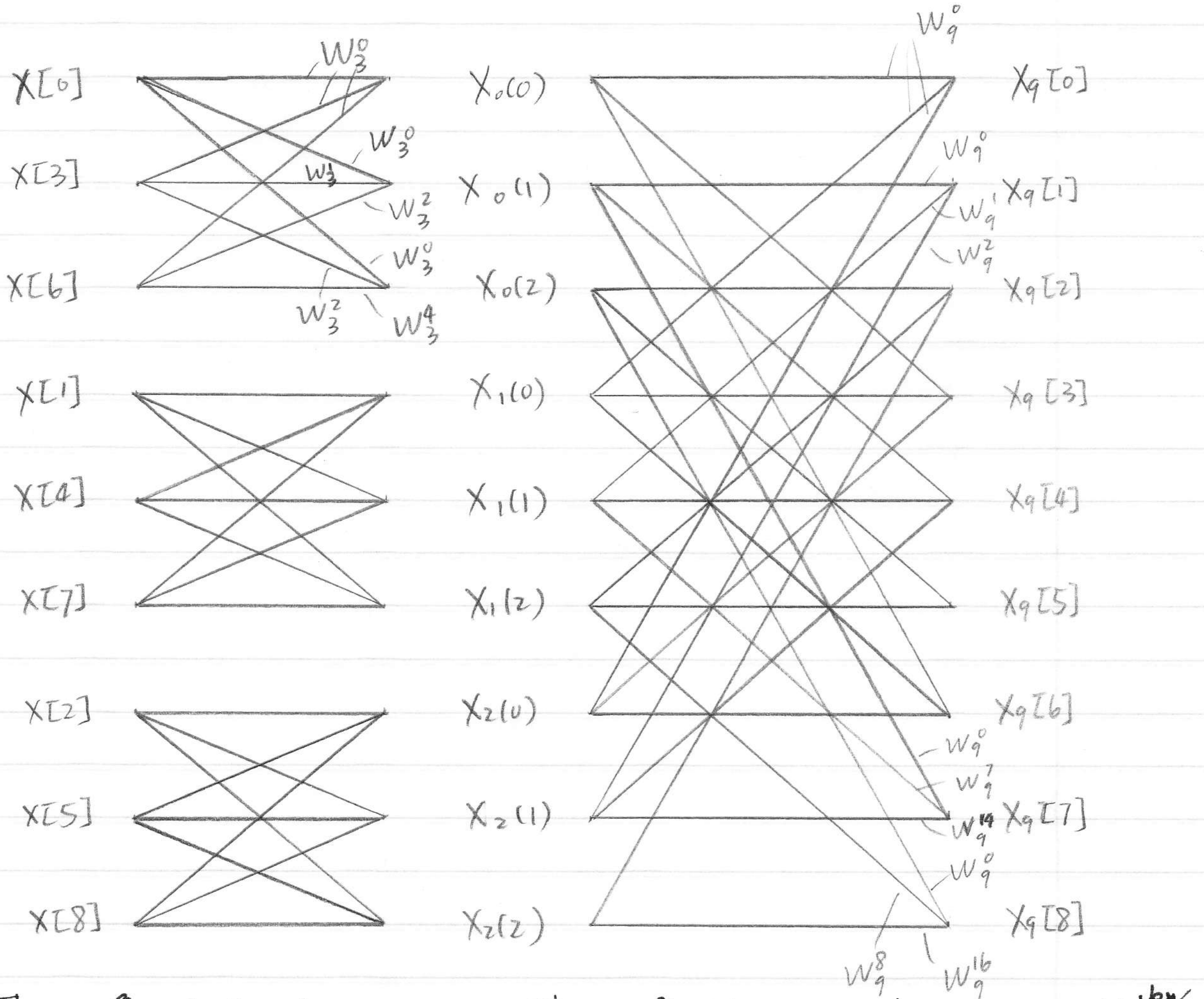
2.

$$X_9[k] = \sum_{n=0}^{8} x[n] e^{-j\frac{2\pi k n}{9}}$$

$$= \sum_{\ell=0}^{2} \sum_{n=0}^{2} x[3\ell+n] e^{-j\frac{2\pi k(3\ell+n)}{9}}$$

$$= \sum_{n=0}^{2} e^{-j\frac{2\pi k n}{9}} \sum_{\ell=0}^{2} x[3\ell+n] e^{-j\frac{2\pi k \ell}{3}}$$

$$= \sum_{n=0}^{2} W_9^{kn} X_n(k \bmod 3)$$



↑ The coefficient for the other 3-point DFT are the same as the first one.

The coefficient on each line is W_9^{kn}, where k is the index of the corresponding output and n is the subindex of $X_n(k \bmod 3)$

3 a)

$$y[n] = \cos(\pi n/3)$$

$$w[n] = u[n] - u[n-9]$$

$$x[n] = y[n]\, w[n]$$

$$Y(w) = \pi\, \mathrm{rep}_{2\pi}[\,\delta(w-w_0) + \delta(w+w_0)\,]$$

$$W(w) = \begin{cases} 9, & w = 0, \pm 2\pi, \ldots \\ \mathrm{psinc}_9(w)\, e^{-j4w}, & \text{otherwise} \end{cases}$$

$$X(w) = Y(w) \circledast W(w)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(w-u)\, W(u)\, du \quad \text{for } |w| \le \pi$$

$$= \frac{1}{2}\left[W\left(w-\frac{\pi}{3}\right) + W\left(w+\frac{\pi}{3}\right)\right]$$

$$= \begin{cases} \frac{9}{2}, & \text{when } w = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi \ldots \text{ or } w = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi \ldots \\ \frac{1}{2}\mathrm{psinc}_9\left(w-\frac{\pi}{3}\right) e^{-j4(w-\frac{\pi}{3})} + \frac{1}{2}\mathrm{psinc}_9\left(w+\frac{\pi}{3}\right) e^{-j4(w+\frac{\pi}{3})}, & \text{otherwise} \end{cases}$$

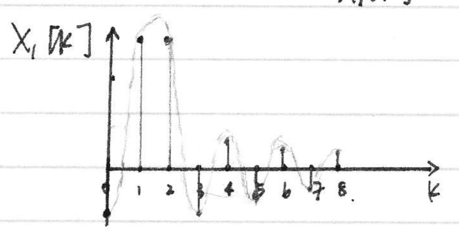
$$X^{(9)}[k] = X(w)\Big|_{w=\frac{2\pi k}{9}} \quad \text{for } k = 0, 1, \ldots, 8.$$

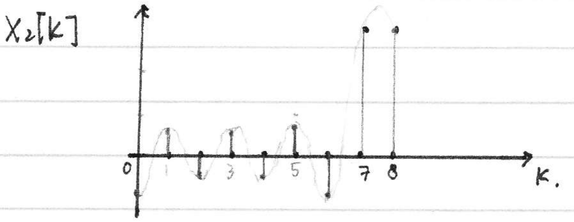
Since there is no integer k satisfying $\frac{2\pi k}{9} = \pm w_0, \pm w_0 \pm 2\pi, \ldots$

$$X^{(9)}[k] = \frac{1}{2}\frac{\sin(\pi(k-\frac{3}{2}))}{\sin(\frac{\pi}{9}(k-\frac{3}{2}))} e^{-j\frac{8}{9}\pi(k-\frac{3}{2})} + \frac{1}{2}\frac{\sin(\pi(k+\frac{3}{2}))}{\sin(\frac{\pi}{9}(k+\frac{3}{2}))} e^{-j\frac{8}{9}\pi(k+\frac{3}{2})}$$

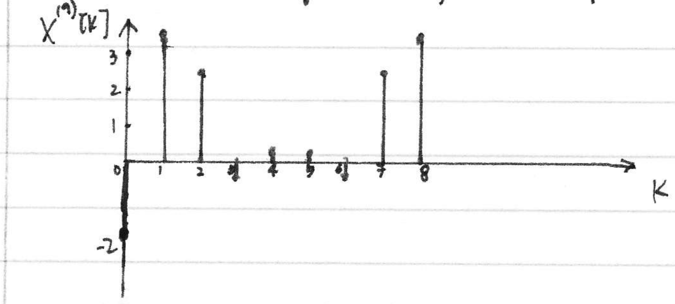
$$= \underbrace{\frac{1}{2}\mathrm{psinc}_9\left(\frac{2\pi k}{9}-\frac{\pi}{3}\right) e^{-j\frac{8}{9}\pi(k-\frac{3}{2})}}_{X_1[k]} + \underbrace{\frac{1}{2}\mathrm{psinc}_9\left(\frac{2\pi k}{9}+\frac{\pi}{3}\right) e^{-j\frac{8}{9}\pi(k+\frac{3}{2})}}_{X_2[k]}$$

b)





Since we could ignore any linear phase terms.



4. (25) Let $x[n], n=0,\ldots,N-1$ denote an N-point signal, where N is assumed to be divisible by 3. Let $X^{(N)}[k], k=0,\ldots,N-1$ denote the N-point Discrete Fourier Transform (DFT) of $x[n]$. Define an $N/3-$point signal $y[n], n=0,\ldots,N/3-1$ according to $y[n]=x[3n], n=0,\ldots,N/3-1$.

Find a simple expression for the $N/3-$point DFT $Y^{(N/3)}[k], k=0,\ldots,N/3-1$ in terms of the N-point DFT $X^{(N)}[k], k=0,\ldots,N-1$.

Approach 1

$$X^{(N)}[k]=\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}$$

$$=\sum_{m=0}^{N/3-1} x[3m]\, e^{-j\frac{2\pi k\cdot 3m}{N}} + \sum_{m=0}^{N/3-1} x[3m+1]\, e^{-j\frac{2\pi k\cdot(3m+1)}{N}} + \sum_{m=0}^{N/3-1} x[3m+2]\, e^{-j\frac{2\pi k\cdot(3m+2)}{N}}$$

$$=\underbrace{\sum_{m=0}^{N/3-1} x[3m]\, e^{-j\frac{2\pi km}{N/3}}}_{Y^{(N/3)}[k]} + e^{-j\frac{2\pi k}{N}}\cdot\underbrace{\sum_{m=0}^{N/3-1} x[3m+1]\, e^{-j\frac{2\pi km}{N/3}}}_{Z^{(N/3)}[k]} + e^{-j\frac{4\pi k}{N}}\cdot\underbrace{\sum_{m=0}^{N/3-1} x[3m+2]\, e^{-j\frac{2\pi km}{N/3}}}_{W^{(N/3)}[k]}$$

for $k=0,1,\cdots,N/3-1$

$$X^{(N)}[k]=Y^{(N/3)}[k]+Z^{(N/3)}[k]\,e^{-j\frac{2\pi k}{N}}+W^{(N/3)}[k]\,e^{-j\frac{4\pi k}{N}} \quad (1)$$

$$X^{(N)}[k+\tfrac{N}{3}]=Y^{(N/3)}[k]+Z^{(N/3)}[k]\,e^{-j\frac{2\pi k}{N}}\cdot e^{-j\frac{2\pi}{3}}+W^{(N/3)}[k]\,e^{-j\frac{4\pi k}{N}}\cdot e^{-j\frac{4\pi}{3}} \quad (2)$$

$$X^{(N)}[k+\tfrac{2N}{3}]=Y^{(N/3)}[k]+Z^{(N/3)}[k]\,e^{-j\frac{2\pi k}{N}}\cdot e^{-j\frac{4\pi}{3}}+W^{(N/3)}[k]\,e^{-j\frac{4\pi k}{N}}\cdot e^{-j\frac{8\pi}{3}} \quad (3)$$

(1)+(2)+(3) $\Rightarrow$

$$X^{(N)}[k]+X^{(N)}[k+\tfrac{N}{3}]+X^{(N)}[k+\tfrac{2N}{3}]=3Y^{(N/3)}[k]$$

$$(\because\ 1+e^{-j\frac{2\pi}{3}}+e^{-j\frac{4\pi}{3}}=1+e^{-j\frac{4\pi}{3}}+e^{-j\frac{8\pi}{3}}=0)$$

$$\Rightarrow Y^{(N/3)}[k]=\frac{1}{3}\left(X^{(N)}[k]+X^{(N)}[k+\tfrac{N}{3}]+X^{(N)}[k+\tfrac{2N}{3}]\right)$$

4. (continued)

Approach 2:

Assume $x[n]$ non-zero only in the range $0 \le n \le N-1$

Then the DFT are related to DTFT by

$$X^{(N)}[k] = X(w)\Big|_{w=\frac{2\pi k}{N}} \qquad (*)$$

$$Y^{(N/3)}[k] = Y(w)\Big|_{w=\frac{2\pi k}{N/3}} \qquad (**)$$

From decimation property

$$Y(w) = \frac{1}{3}\sum_{m=0}^{2} X\left(\frac{w-2\pi m}{3}\right) = \frac{1}{3}\left[X\left(\frac{w}{3}\right) + X\left(\frac{w}{3}-\frac{2\pi}{3}\right) + X\left(\frac{w}{3}-\frac{4\pi}{3}\right)\right]$$

From $(*)$, $(**)$

$$Y^{(N/3)}[k] = Y\left(\frac{2\pi k}{N/3}\right) = \frac{1}{3}\left[X\left(\frac{2\pi k}{N}\right) + X\left(\frac{2\pi k}{N}-\frac{2\pi}{3}\right) + X\left(\frac{2\pi k}{N}-\frac{4\pi}{3}\right)\right]$$

$$= \frac{1}{3}\left[X\left(\frac{2\pi k}{N}\right) + X\left(\frac{2\pi k}{N}+\frac{4\pi}{3}\right) + X\left(\frac{2\pi k}{N}+\frac{2\pi}{3}\right)\right]$$ $\because$ DTFT (periodic with 2π)

$$= \frac{1}{3}\left[X\left(\frac{2\pi k}{N}\right) + X\left(\frac{2\pi}{N}\left(k+\frac{2N}{3}\right)\right) + X\left(\frac{2\pi}{N}\left(k+\frac{N}{3}\right)\right)\right]$$

$$= \frac{1}{3}\left[X^{(N)}[k] + X^{(N)}\left[k+\frac{2N}{3}\right] + X^{(N)}\left[k+\frac{N}{3}\right]\right]$$

where $k = 0, 1, \cdots, N/3 - 1$

4. Approach 3.

$$Y^{(N/3)}[k] = \sum_{n=0}^{N/3-1} x[3n]\, e^{-j\frac{2\pi k n}{N/3}}$$

$$x[3n] = \frac{1}{N}\sum_{l=0}^{N-1} X^{(N)}[l]\, e^{j\frac{2\pi l \cdot 3n}{N}}$$

$$Y^{(N/3)}[k] = \sum_{n=0}^{N/3-1} \frac{1}{N}\sum_{l=0}^{N-1} X^{(N)}[l]\, e^{j\frac{2\pi l \cdot n}{N/3}} \cdot e^{-j\frac{2\pi k n}{N/3}}$$

$$= \sum_{l=0}^{N-1} X^{(N)}[l] \cdot \frac{1}{N} \underbrace{\sum_{n=0}^{N/3-1} e^{j\frac{2\pi (l-k) n}{N/3}}}$$

$$= \begin{cases} \frac{N}{3}, & \text{when } l-k \text{ is multiples of } \frac{N}{3} \\ 0, & \text{otherwise} \end{cases}$$

Since $0 \le l \le N-1$,

$0 \le k \le N/3 - 1$

the non-zero values occur at $l - k = 0, \frac{N}{3}, \frac{2N}{3}$

$$= \sum_{l=0}^{N-1} X^{(N)}[l] \cdot \frac{1}{N} \cdot \frac{N}{3}\left[\delta[l-k] + \delta[l-(k+\tfrac{N}{3})] + \delta[l-(k+\tfrac{2N}{3})]\right]$$

$$= \frac{1}{3}\left(X^{(N)}[k] + X^{(N)}[k+\tfrac{N}{3}] + X^{(N)}[k+\tfrac{2N}{3}]\right), \quad 0 \le k \le N/3-1$$