13 February 2013 **Name:** Solution

ECE 438 **Exam No. 1** **Spring 2013**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n]=\frac{1}{2}x[n]+x[n-1]+\frac{1}{2}x[n-2]$$

a. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.

b. (10) Based on your answer to part (a), find simple expressions for the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response of this system.

c. (5) Based on your answer to part (b), carefully sketch the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response of this system.

Answer:

a. Consider $x(n)=e^{j\omega n}$, ω is fixed.

$$H(\omega)=y_\omega(0).$$

$$=\frac{1}{2}+e^{-j\omega}+\frac{1}{2}e^{-j2\omega}$$

$$=e^{-j\omega}\left(\frac{1}{2}e^{j\omega}+1+\frac{1}{2}e^{-j\omega}\right)$$

$$=e^{-j\omega}\left[1+\frac{1}{2}\left(e^{j\omega}+e^{-j\omega}\right)\right]$$

$$=\underline{\underline{e^{-j\omega}(1+\cos\omega)}}$$

b. Based on part (a),

$$|H(\omega)|=|e^{-j\omega}(1+\cos\omega)|=|1+\cos\omega|$$

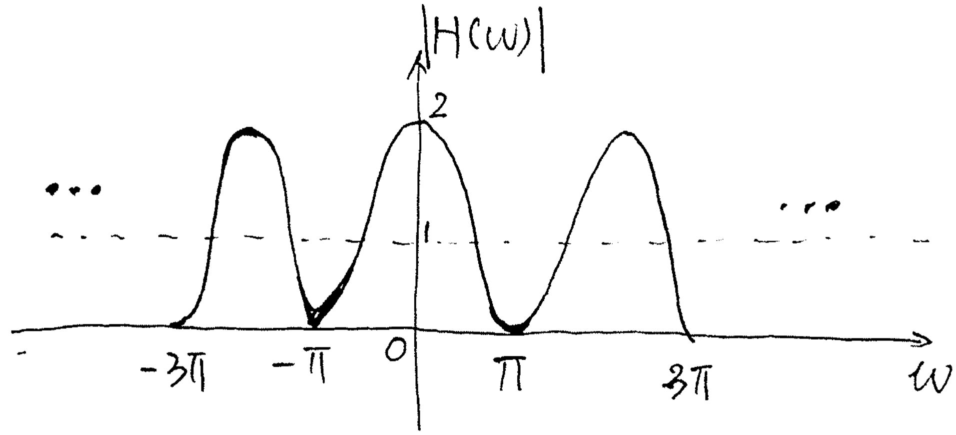
$\because 1+\cos\omega\geq 0 \quad \therefore |1+\cos\omega|=1+\cos\omega.$

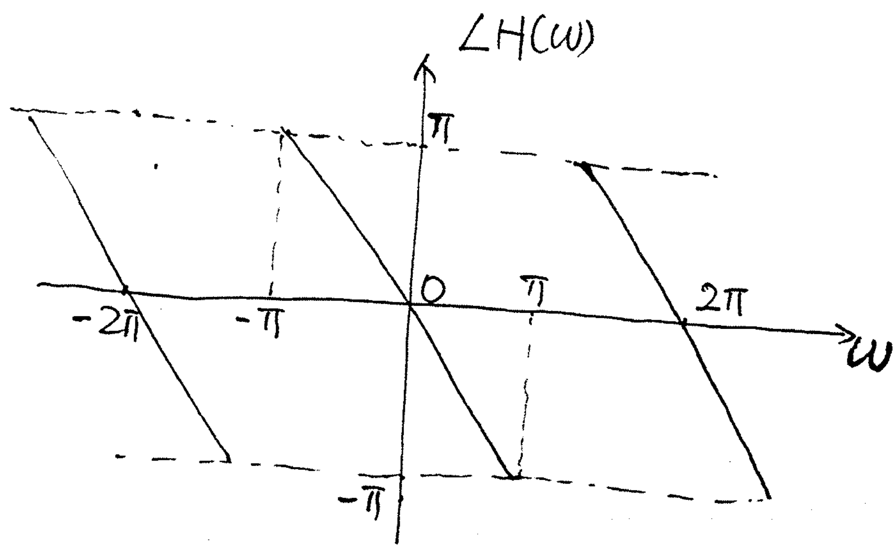
$\therefore |H(\omega)|=\underline{1+\cos\omega}.$

1. (continued)

$$\angle H(\omega) = \angle e^{-j\omega} + \angle(1+\cos(\omega))$$
$$= -\omega + 0$$
$$= \underline{\underline{-\omega}}$$

c.





2. $x[n] = \cos\left(\frac{\pi n}{4}\right)\left(u[n-16] - u[n]\right)$

$$\mathrm{DTFT}\left(\cos\left(\tfrac{\pi}{4}n\right)\right) = \pi\left(\delta\left(w-\tfrac{\pi}{4}\right) + \delta\left(w+\tfrac{\pi}{4}\right)\right)$$

$$\mathrm{DTFT}\left(u[n-16] - u[n]\right) = \sum_{n=0}^{15} e^{-jwn} = \frac{1-e^{-j16w}}{1-e^{-jw}} = \frac{e^{-j8w}\left(e^{j8w} - e^{-j8w}\right)}{e^{-j\frac{w}{2}}\left(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\right)}$$

$$= e^{-j\frac{15}{2}w}\frac{\sin 8w}{\sin\frac{w}{2}}$$

$$\Rightarrow X(w) = \pi\left(\delta\left(w-\tfrac{\pi}{4}\right) + \delta\left(w+\tfrac{\pi}{4}\right)\right) * e^{-j\frac{15}{2}w}\frac{\sin 8w}{\sin\frac{w}{2}}$$

$$= \pi e^{-j\frac{15}{2}(w-\frac{\pi}{4})}\frac{\sin 8(w-\frac{\pi}{4})}{\sin\frac{1}{2}(w-\frac{\pi}{4})} + \pi e^{-j\frac{15}{2}(w+\frac{\pi}{4})}\frac{\sin 8(w+\frac{\pi}{4})}{\sin\frac{1}{2}(w+\frac{\pi}{4})}$$

$$= \pi e^{-j\frac{15}{2}(w-\frac{\pi}{4})}\frac{\sin 8w}{\sin\frac{1}{2}(w-\frac{\pi}{4})} + \pi e^{-j\frac{15}{2}(w+\frac{\pi}{4})}\frac{\sin 8w}{\sin\frac{1}{2}(w+\frac{\pi}{4})}$$

$$= \pi \sin 8w\, e^{-j\frac{15}{2}w}\left(\frac{e^{j\frac{15}{8}\pi}}{\sin\frac{1}{2}(w-\frac{\pi}{4})} + \frac{e^{-j\frac{15}{8}\pi}}{\sin\frac{1}{2}(w+\frac{\pi}{4})}\right)$$

$$= \pi \sin 8w e^{-j\frac{15}{2}w}\frac{1}{\sin\frac{1}{2}(w-\frac{\pi}{4})\sin\frac{1}{2}(w+\frac{\pi}{4})}\cdot\frac{1}{2j}\left[e^{j\frac{15}{8}\pi}\left(e^{j(\frac{w}{2}+\frac{\pi}{8})} - e^{-j(\frac{w}{2}+\frac{\pi}{8})}\right) + e^{-j\frac{15}{8}\pi}\left(e^{j(\frac{w}{2}-\frac{\pi}{8})} - e^{-j(\frac{w}{2}-\frac{\pi}{8})}\right)\right]$$

$$= \pi \sin 8w e^{-j\frac{15}{2}w}\frac{\left(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\cdot\frac{1}{\sqrt{2}}\right)}{\sin\frac{1}{2}(w-\frac{\pi}{4})\sin\frac{1}{2}(w+\frac{\pi}{4})\cdot j}$$

$$|X(w)| = \left|\pi \sin 8w\frac{\left(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\frac{1}{\sqrt{2}}\right)}{\sin\frac{1}{2}(w-\frac{\pi}{4})\sin\frac{1}{2}(w+\frac{\pi}{4})}\right|$$

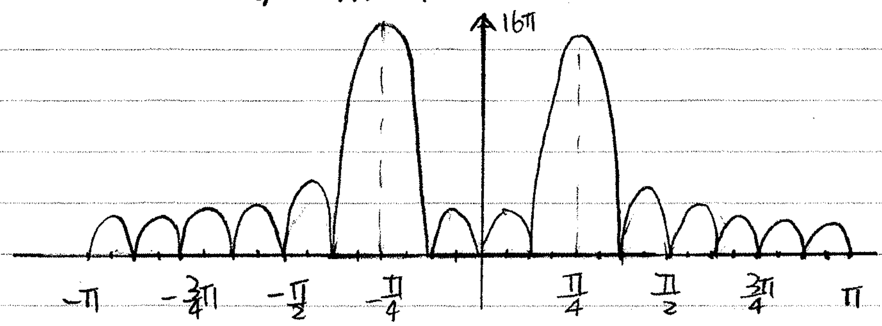
First, check the points where $|X(w)| = 0$

Since $\left|e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\frac{1}{\sqrt{2}}\right| \neq 0$, only need to check

if $\sin 8w \cdot \frac{1}{\sin\frac{1}{2}(w-\frac{\pi}{4})\sin\frac{1}{2}(w+\frac{\pi}{4})} = 0$

$\Rightarrow w = \pm\frac{\pi}{8}, \pm\frac{3}{8}\pi, \pm\frac{\pi}{2}, \pm\frac{5}{8}\pi, \pm\frac{3}{4}\pi, \pm\frac{7}{8}\pi, \pm\pi$

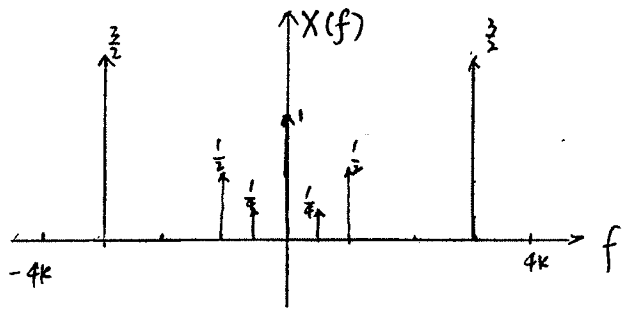
When $w = \pm\frac{\pi}{4}$, $|X(w)| = \pi\cancel{\sin(2\pi)} \cdot 16\pi$



3. $\quad x(t) = 1 + \frac{1}{2}\cos(2\pi\cdot 500t) + \cos(2\pi\cdot 1000t) + 3\cos(2\pi\cdot 3000t)$

$x(t) \xrightarrow{\text{CTFT}} X(f)$

$$X(f) = \delta(f) + \tfrac{1}{4}\delta(f-500) + \tfrac{1}{4}\delta(f+500) + \tfrac{1}{2}\delta(f-1000) + \tfrac{1}{2}\delta(f+1000) + \tfrac{3}{2}\delta(f-3000) + \tfrac{3}{2}\delta(f+3000)$$



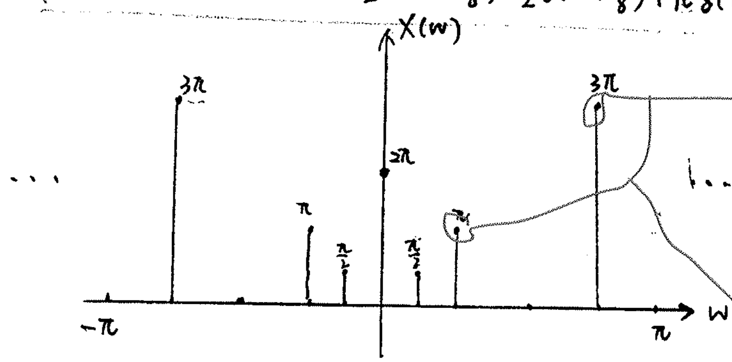
(a) $\quad x[n] = x\left(\frac{n}{f_s}\right)$, $\quad X_d(\omega) = f_s\, \text{rep}_{f_s}\left[X\left(\frac{\omega}{2\pi} f_s\right)\right]$

$$\Rightarrow X(\omega) = f_s\, \text{rep}_{f_s}\left[\delta\left(\tfrac{\omega}{2\pi}f_s\right) + \tfrac{1}{4}\delta\left(\tfrac{\omega f_s}{2\pi}-500\right) + \tfrac{1}{4}\delta\left(\tfrac{\omega f_s}{2\pi}+500\right) + \tfrac{1}{2}\delta\left(\tfrac{\omega f_s}{2\pi}-1000\right) + \tfrac{1}{2}\delta\left(\tfrac{\omega f_s}{2\pi}+1000\right) + \tfrac{3}{2}\delta\left(\tfrac{\omega f_s}{2\pi}-3000\right) + \tfrac{3}{2}\delta\left(\tfrac{\omega f_s}{2\pi}+3000\right)\right]$$

$\delta\left(\frac{\omega f_s}{2\pi}\right) = \frac{2\pi}{f_s}\delta(\omega)$. therefore,

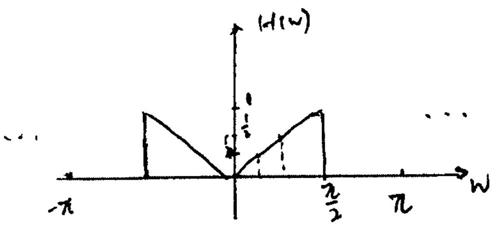
$$X(\omega) = \text{rep}_{2\pi}\left[2\pi\delta(\omega) + \tfrac{\pi}{2}\delta\left(\omega - \tfrac{1000\pi}{f_s}\right) + \tfrac{\pi}{2}\delta\left(\omega + \tfrac{1000\pi}{f_s}\right) + \tfrac{1}{2}\delta\left(\omega - \tfrac{2000\pi}{f_s}\right) + \tfrac{1}{2}\delta\left(\omega + \tfrac{2000\pi}{f_s}\right) + \tfrac{3}{2}\delta\left(\omega - \tfrac{6000\pi}{f_s}\right) + \tfrac{3}{2}\delta\left(\omega + \tfrac{6000\pi}{f_s}\right)\right]$$

$$\Rightarrow \boxed{X(\omega) = \text{rep}_{2\pi}\left[2\pi\delta(\omega) + \tfrac{\pi}{2}\delta\left(\omega - \tfrac{\pi}{8}\right) + \tfrac{\pi}{2}\delta\left(\omega + \tfrac{\pi}{8}\right) + \pi\delta\left(\omega - \tfrac{\pi}{4}\right) + \pi\delta\left(\omega + \tfrac{\pi}{4}\right) + 3\pi\delta\left(\omega - \tfrac{3}{4}\pi\right) + 3\pi\delta\left(\omega + \tfrac{3}{4}\pi\right)\right]}$$

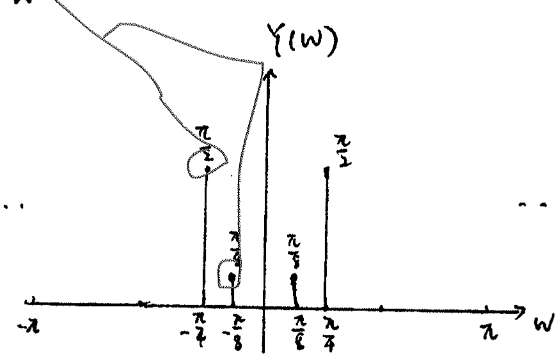


These are still continuous-parameter impulses; so the heads should be arrows, as in your figure for X(f).

(b) $\quad Y(\omega) = X(\omega)\cdot H(\omega)$



$\Rightarrow$



$$\boxed{Y(\omega) = \text{rep}_{2\pi}\left[\tfrac{\pi}{8}\delta\left(\omega - \tfrac{\pi}{8}\right) + \tfrac{\pi}{8}\delta\left(\omega + \tfrac{\pi}{8}\right) + \tfrac{\pi}{2}\delta\left(\omega - \tfrac{\pi}{4}\right) + \tfrac{\pi}{2}\delta\left(\omega + \tfrac{\pi}{4}\right)\right]}$$

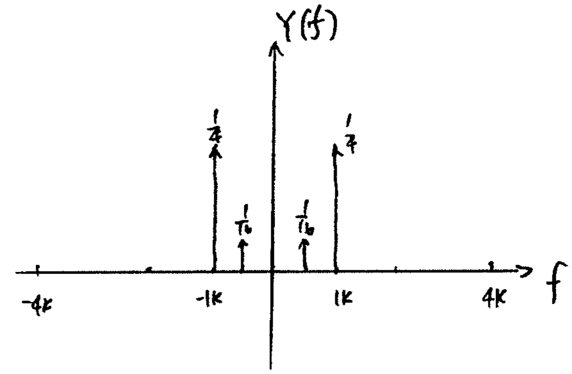
(c) $y[n] \longrightarrow \boxed{\underset{f_s = 8\text{kHz}}{\text{ideal D/A}}} \longrightarrow y(t)$

Let $Y(f) = \mathcal{F}\{y(t)\}$

Then $Y(f) = Y(\omega)\big|_{\omega = \frac{2\pi f}{f_s}}$ and filtered with LPF of cut-off frequency $\frac{f_s}{2} = 4\text{kHz}$.

$$\Rightarrow Y(f) = \frac{1}{16}\delta(f-500) + \frac{1}{16}\delta(f+500) + \frac{1}{4}\delta(f-1000) + \frac{1}{4}\delta(f+1000)$$

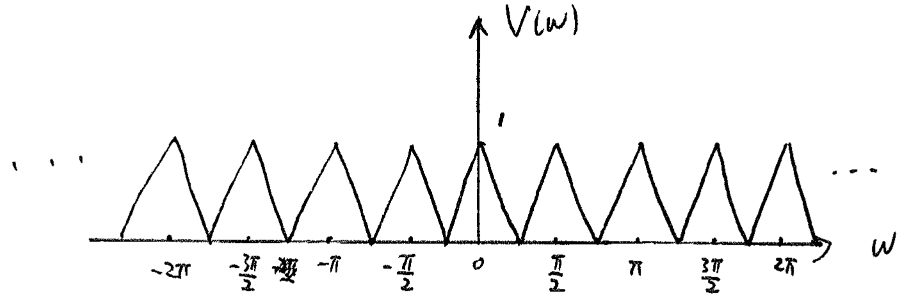
$$\Rightarrow \boxed{y(t) = \frac{1}{8}\cos(2\pi\cdot 500t) + \frac{1}{2}\cos(2\pi\cdot 1000t).}$$



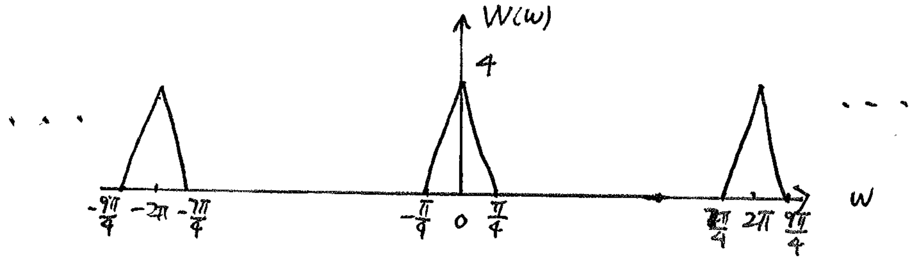
4. (continued - 1)

a) $V(w) = X(4w)$

b)



c) $W(w) = V(w)H(w) = X(4w)H(w)$



d)

e) $Y(w) = \frac{1}{4}\sum_{k=0}^{3} W\left(\frac{w-2\pi k}{4}\right) = \frac{1}{4}\sum_{k=0}^{3} X(w)\, H\left(\frac{w-2\pi k}{4}\right)$

Note: $X(4(w-2\pi k)/4) = X(w)$

f)