3 May 2012 **Name:** Solution **Signature:** ________________

ECE 438 **Final Exam** **Spring 2012**

- You have 120 minutes to work the following five problems.
- Be sure to show **all** your work to obtain full credit.
- You do *not* need to derive any result that can be found on the formula sheet. However, you should state that it can be found there.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.
- It will be to your advantage to budget your time so that you can write something for each problem.

1. (25 pts.) Consider a system initially at rest, described by the following equation

$$y[n] = x[n] - x[n-1] - y[n-1].$$

a. (7) Find the response $y[n]$ to the following input

$$x[n] = \begin{cases} (-1)^n, & 0 \le n \le 4 \\ 0, & n < 0, 5 \le n \end{cases}$$

b. (6) Find a simple expression for the frequency response $H(\omega)$ for this system.

c. (5) From your answer to part (b), determine simple expressions for the magnitude and phase of the frequency response.

d. (5) Find the response to the input $x[n] = \cos(\pi n / 3)$ using your answer to part (c). Do **not** use the system equation given at the beginning of the problem statement.

e. (2) Is this system stable or unstable? Explain why or why not.

a)

n	$y[n]$	$y[n-1]$	$x[n]$	$x[n-1]$
0	1	0	1	0
1	-3	1	-1	1
2	5	-3	1	-1
3	-7	5	-1	1
4	9	-7	1	-1
5	-10	9	0	1
6	10	-10	0	0
7	-10	10	0	0
⋮	⋮	⋮	⋮	⋮

$$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, -3, 5, -7, 9, -10, 10, -10, \dots\}$$

1. (continued – 1)

b) Find $H(\omega)$:

$$y[n] = x[n] - x[n-1] - y[n-1]$$
$$Y(\omega) = X(\omega) - X(\omega)e^{-j\omega} - Y(\omega)e^{-j\omega}$$
$$Y(\omega)(1+e^{-j\omega}) = X(\omega)(1-e^{-j\omega})$$
$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})}{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}})}$$
$$= j\frac{\sin(\frac{\omega}{2})}{\cos(\frac{\omega}{2})} = j\tan(\frac{\omega}{2})$$

c) $|H(\omega)| = |\tan(\frac{\omega}{2})|$

$$\angle H(\omega) = \angle j\tan(\tfrac{\omega}{2}) = \begin{cases} \frac{\pi}{2} & 0 < \omega < \pi \\ -\frac{\pi}{2} & -\pi < \omega < 0 \end{cases}$$

d) If $x[n] = e^{j\omega_0 n}$, $y[n] = H(\omega_0)e^{j\omega_0 n}$

$$x[n] = \cos(\tfrac{\pi}{3}n) = \tfrac{1}{2}e^{j\frac{\pi}{3}n} + \tfrac{1}{2}e^{-j\frac{\pi}{3}n}$$
$$y[n] = \tfrac{1}{2}H(\tfrac{\pi}{3})e^{j\frac{\pi}{3}n} + \tfrac{1}{2}H(-\tfrac{\pi}{3})e^{-j\frac{\pi}{3}n}$$
$$= \frac{1}{2}\cdot j\frac{\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{6})}e^{j\frac{\pi}{3}n} + \frac{1}{2}j\frac{\sin(-\frac{\pi}{6})}{\cos(-\frac{\pi}{6})}e^{-j\frac{\pi}{3}n}$$
$$= j\frac{\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{6})}\left(\frac{1}{2}e^{j\frac{\pi}{3}n} - \frac{1}{2}e^{-j\frac{\pi}{3}n}\right)$$
$$= -\tan(\tfrac{\pi}{6})\sin(\tfrac{\pi}{3}n) = -\frac{1}{\sqrt{3}}\sin(\tfrac{\pi}{3}n)$$

e) No, this is not BIBO stable.
The impulse response is $h[n] = \{1, -1, 1, -1, \cdots\}$.
Because $\sum |h[n]| = \infty$, this system is not stable.

2. (25 pts.) The output $y[n]=\left(\frac{1}{2}\right)^n u[n]$ is observed from a discrete-time, linear, time-invariant system described by the system equation $y[n]=x[n]-x[n-1]+\frac{1}{2}y[n-1]$. Using ZT techniques, find the input $x[n]$ that produced this output.

$$Y(z)=X(z)-X(z)z^{-1}+\frac{1}{2}Y(z)z^{-1}$$

$$\Rightarrow \quad X(z)=\frac{(1-\frac{1}{2}z^{-1})\,Y(z)}{1-z^{-1}}$$

$$y[n]=\left(\tfrac{1}{2}\right)^n u(n) \overset{ZT}{\Longrightarrow} Y(z)=\frac{1}{1-\frac{1}{2}z^{-1}}$$

So $\quad X(z)=\dfrac{1}{1-z^{-1}}$

$\overset{ZT^{-1}}{\Longrightarrow}$ either $\boxed{x_1(n)=u(n)}$ or $\boxed{x_2(n)=-u(-n-1)}$

produces $y[n]=\left(\frac{1}{2}\right)^n u(n)$

Actually, by linearity $\quad a x_1(n)+b x_2(n) \rightarrow a y(n)+b y(n)$
$= (a+b)\,y(n)$

So any input of the form,

$$\boxed{x(n)=a\,u(n)-b\,u(-n-1), \quad \text{where } a+b=1}$$

will produce $y(n)=\left(\frac{1}{2}\right)^n u(n)$.

Note this general form doesn't have a valid Z-transform, in general, therefore we had to find it by taking the left and right-sided cases separately.

3. (25) Consider the DT signal $x[n]$ defined for non-negative time $n \geq 0$ according to:

$$x[n] = \begin{cases} \sin(\pi n / 4), & n = 0,1,\ldots,7, \quad 16,17,\ldots,23, \quad 32,33,\ldots,39, \quad \ldots \\ 0, & n = 8,9,\ldots 15, \quad 24,25,\ldots,31, \quad 40,41,\ldots,47, \quad \ldots \end{cases}$$

a. (6) Carefully sketch $x[n]$ for $0 \leq n \leq 31$

Suppose we consider $x[n]$ as a model for a particular speech utterance. Based on this viewpoint, answer parts (b) through (e) below:

b. (1) Is the utterance voiced or unvoiced?

c. (1) Find the pitch period P' in samples.

d. (2) Carefully sketch the DT vocal tract response $v[n]$.

e. (1) Find the formant frequencies ω_i in radians/sample.

Suppose we compute the STDTFT $X(\omega, n)$ of this waveform according to:

$$X(\omega, n) = \sum_k x[k] w[n-k] e^{-j\omega k},$$

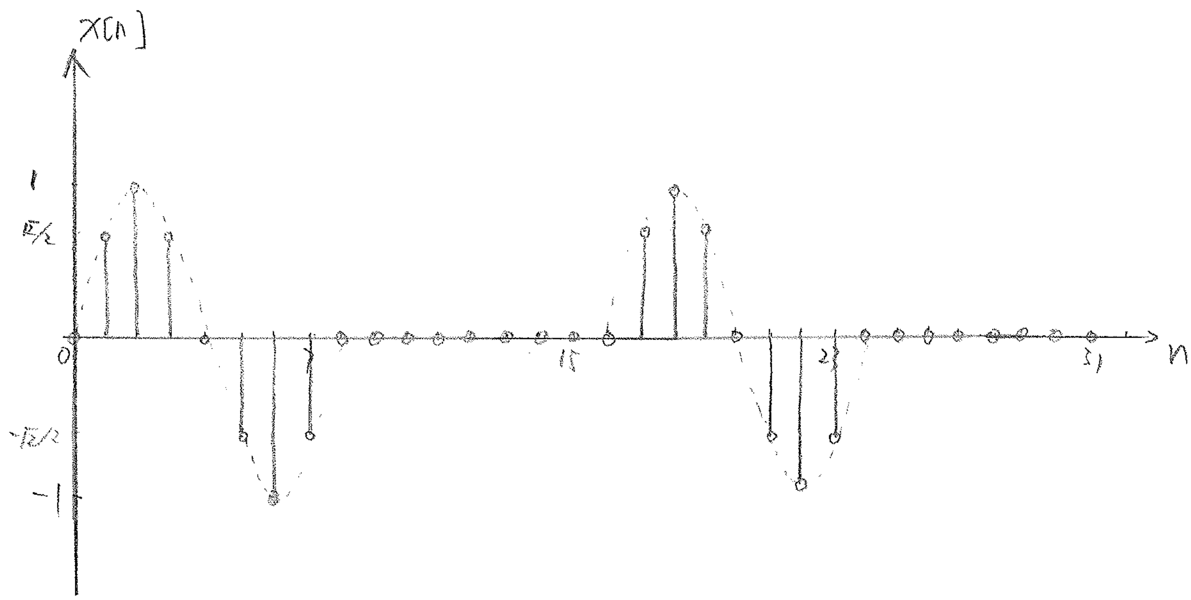
where the window function is defined as $w[k] = \begin{cases} 1, & k = 0,1,2,\ldots,7 \\ 0, & \text{else} \end{cases}$.

f. (12) Compute $X(\omega, n)$ for $n = 7$. Simplify your answer as much as possible.

g. (1) Compute $X(\omega, n)$ for $n = 15$. Simplify your answer as much as possible.

h. (1) Is this spectrogram wideband or narrowband?

a.



3. (continued – 1)

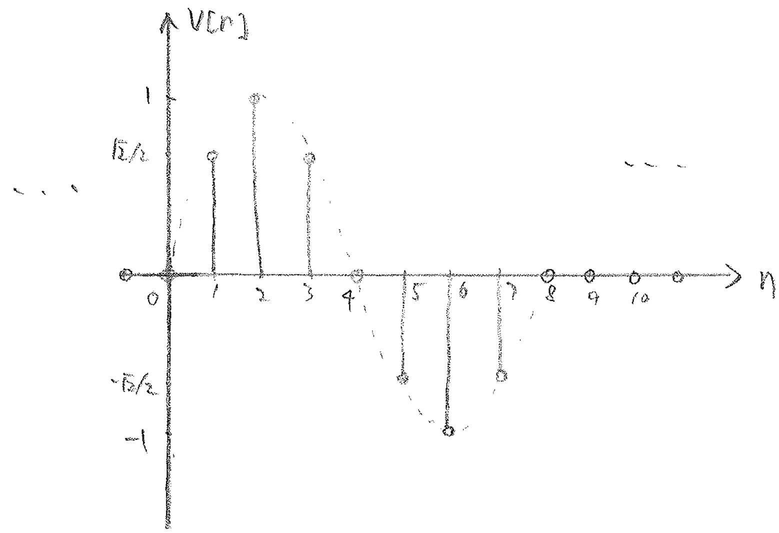
b. the utterance is voiced

c. p' is 16 samples

d. $x[n] = V[n] * e[n]$

in this case $e[n] = \sum_{k=-\infty}^{\infty} \delta[n-16k]$

$$V[n] = \begin{cases} \sin\left(\frac{\pi n}{4}\right), & n = 0, 1, \cdots, 6, 7 \\ 0, & \text{else} \end{cases} = \sin\left(\frac{\pi n}{4}\right)\left(u[n] - u[n-8]\right)$$



e. $\omega_i = \frac{\pi}{4}$ radians/sample

3.

f. when $n=7$

$$w[n-k] = \begin{cases} 1, & k=0,1,2,\cdots,7 \\ 0, & \text{else} \end{cases}$$

$$x[k] = \sin\left(\frac{\pi n}{4}\right), \quad k=0,1,2,\cdots,7$$

then $w[n-k] \overset{DTFT}{\longleftrightarrow} W(\omega) = \sum_{k=0}^{7} 1 \cdot e^{-j\omega k} = e^{-j\frac{7}{2}\omega} \dfrac{\sin 4\omega}{\sin\frac{\omega}{2}}$ (periodic with 2π)

$$x[k] \overset{DTFT}{\longleftrightarrow} X(\omega) = \frac{\pi}{j} \, rep_{2\pi}\left[\delta(\omega-\tfrac{\pi}{4}) - \delta(\omega+\tfrac{\pi}{4})\right]$$

Applying DTFT product relation

$$x[n]y[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mu)Y(\omega-\mu)\,d\mu$$

$$X(\omega,7) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mu)W(\omega-\mu)\,d\mu$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\pi}{j}\left[\delta(\mu-\tfrac{\pi}{4}) - \delta(\mu+\tfrac{\pi}{4})\right] W(\omega-\mu)\,d\mu$$

$$= \frac{1}{2j} W(\omega-\tfrac{\pi}{4}) - \frac{1}{2j} W(\omega+\tfrac{\pi}{4})$$

$$= \frac{1}{2j} \frac{\sin 4(\omega-\frac{\pi}{4})}{\sin\frac{1}{2}(\omega-\frac{\pi}{4})} \cdot e^{-j\frac{7}{2}(\omega-\frac{\pi}{4})} - \frac{1}{2j} \cdot \frac{\sin 4(\omega+\frac{\pi}{4})}{\sin\frac{1}{2}(\omega+\frac{\pi}{4})} \cdot e^{-j\frac{7}{2}(\omega+\frac{\pi}{4})}$$

g. when $n=15$

$$w[n-k] = \begin{cases} 1, & k=8,9,\cdots,15 \\ 0, & \text{else} \end{cases}$$

$$x[k] = 0, \quad k=8,9,\cdots,15$$

$$X(\omega,15) = 0$$

h. the window length is comparable to one pitch period

the spectrogram is wideband

Alternative solution for 3f.

Compute $X(w,7)$ by definition

$$X(w,7) = \sum_k x[k]w[7-k]e^{-jwk}$$

$$= \sum_{k=0}^{7} x[k]\cdot 1 \cdot e^{-jwk}$$

$$= \sum_{k=0}^{7} \sin\left(\frac{\pi k}{4}\right)e^{-jwk}$$

$$= \sum_{k=0}^{7} \frac{1}{2j}\left(e^{j\frac{\pi}{4}k} - e^{-j\frac{\pi}{4}k}\right)e^{-jwk}$$

$$= \frac{1}{2j}\sum_{k=0}^{7} e^{-j(w-\frac{\pi}{4})k} - \frac{1}{2j}\sum_{k=0}^{7} e^{-j(w+\frac{\pi}{4})k}$$

$$= \frac{1}{2j}\cdot\frac{1-e^{-j(w-\frac{\pi}{4})8}}{1-e^{-j(w-\frac{\pi}{4})}} - \frac{1}{2j}\frac{1-e^{-j(w+\frac{\pi}{4})8}}{1-e^{-j(w+\frac{\pi}{4})}}$$

$$= \frac{1}{2j}\cdot\frac{\sin 4(w-\frac{\pi}{4})}{\sin\frac{1}{2}(w-\frac{\pi}{4})}\cdot e^{-j\frac{7}{2}(w-\frac{\pi}{4})} - \frac{1}{2j}\cdot\frac{\sin 4(w+\frac{\pi}{4})}{\sin\frac{1}{2}(w+\frac{\pi}{4})}\cdot e^{-j\frac{7}{2}(w+\frac{\pi}{4})}$$

4. (25 pts) Consider the 2-D function $f(x,y) = \cos\left[2\pi(4x + y)\right]$.

a. (5) Sketch $f(x,y)$ with enough detail to show that you know what it looks like. Be sure to dimension your axes.

b. (4) Find a simple expression for the continuous-space Fourier transform (CSFT) $F(u,v)$ of $f(x,y)$, and sketch $F(u,v)$. Be sure to dimension your axes.

Suppose that we sample $f(x,y)$ at the resolution of 5 samples/inch in both the x and y directions, using the 2-D comb operator, to yield $f_s(x,y)$.

c. (4) Find an expression for $F_s(u,v)$ in terms of the 2-D rep operator. Sketch $F_s(u,v)$ with enough detail to show that you know what it looks like. Be sure to dimension your axes.

Now suppose that we reconstruct a continuous-parameter image $f_r(x,y)$ by convolving $f_s(x,y)$ with a 2-D filter that has point-spread function $h(x,y) = \text{sinc}(5x,5y)$.

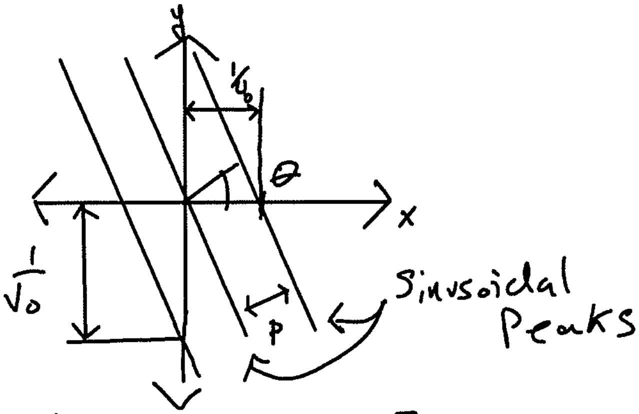
d. (9) Find a simple expression for $f_r(x,y)$ that does not contain any operators. *Hint:* To successfully complete this part, you must work in the frequency domain.

e. (3) Sketch $f_r(x,y)$ with enough detail to show that you know what it looks like. Be sure to dimension your axes.

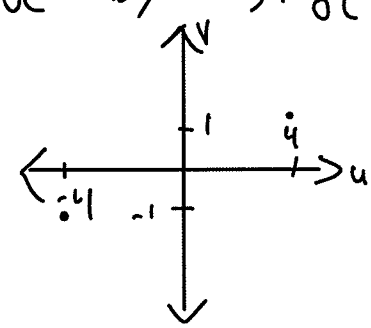
a) $f(x,y) = \cos(2\pi(u_0 x + v_0 y))\Big|_{u_0=4,\ v_0=1} = \cos(2\pi(4x+y))$

$p = \dfrac{1}{\sqrt{u_0^2+v_0^2}} = \dfrac{1}{\sqrt{17}}$

$\theta = \text{atan}\left(\frac{v_0}{u_0}\right) = 0.245$ rad

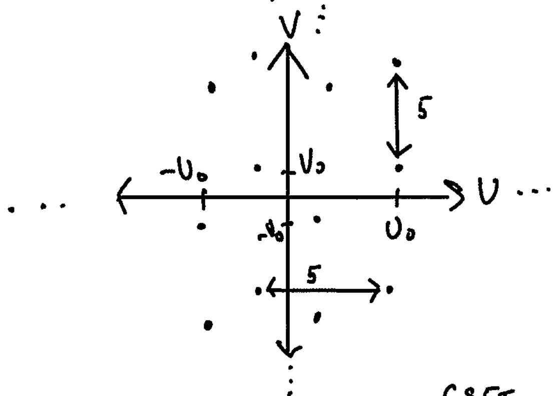


b) $F(u,v) = \frac{1}{2}\left[\delta(u-u_0, v-v_0) + \delta(u+u_0, v+v_0)\right]$



4. (continued – 1)

c) $\text{comb}_{\frac{1}{5},\frac{1}{5}}\left[f(x,y)\right] \overset{CSFT}{\Longleftrightarrow} 25\,\text{rep}_{5,5}\left[F(u,v)\right]$



$25\,\text{rep}_{5,5}\left[\frac{1}{2}\left(\delta(u-u_0, v-v_0)+\delta(u+u_0, v+v_0)\right)\right]$

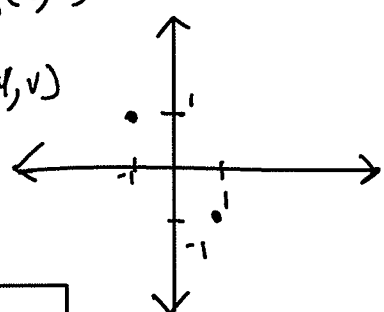
d) $\text{sinc}(5x, 5y) \overset{CSFT}{\Longleftrightarrow} \frac{1}{25}\,\text{rect}\left(\frac{1}{5}u, \frac{1}{5}v\right)$

$F_r(u,v) = \frac{1}{25}\,\text{rect}\left(\frac{1}{5}u, \frac{1}{5}v\right) F_s(u,v)$

$F_r(u,v)$

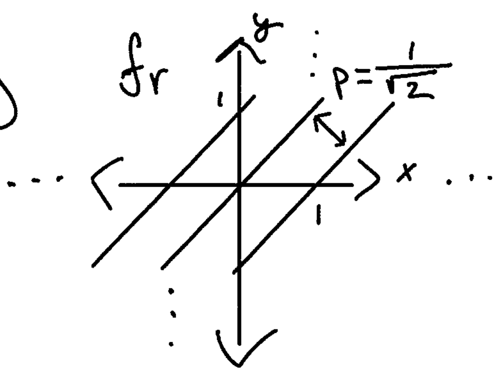
$\frac{1}{5}u = \frac{1}{2}$

$u = 5/2$



$$\Rightarrow \boxed{f_r(x,y) = \cos(2\pi(x-y))}$$

e)



5. (25 pts) Consider a spatial filter with point spread function $h[m,n] = h_{\text{ver}}[m] \cdot h_{\text{hor}}[n]$ given below

$$h_{\text{ver}}[m] = \begin{cases} 1, & m = 0 \\ \frac{1}{2}, & m = -1, +1 \\ 0, & \text{else} \end{cases} \qquad h_{\text{hor}}[n] = \begin{cases} \frac{3}{2}, & n = 0 \\ -\frac{1}{2}, & n = -1, +1 \\ 0, & \text{else} \end{cases}$$

a. (10) Find the output $g[m,n]$ when this filter is applied to the following input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 11×11 set of pixels in the input image.

0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

b. (12) Find a simple expression for the frequency response $H(\mu, \nu)$ of this filter, and sketch the magnitude $|H(\mu, \nu)|$ along the μ axis, the ν axis and the $\mu = \nu$ axis.

c. (3) Using your results from parts a) and b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

5. (continued – 1)

a)

$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	0	0
$-\frac{3}{4}$	$6/4$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$5/4$	0	$-\frac{1}{4}$	0	0	0
-1	2	1	1	1	$5/4$	1	0	$-\frac{1}{4}$	0	0
-1	2	1	1	1	1	$5/4$	1	0	$-\frac{1}{4}$	0
-1	2	1	1	1	1	1	$5/4$	1	0	$-\frac{1}{4}$
-1	2	1	1	1	1	1	1	$\frac{6}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$
-1	2	1	1	1	1	1	$5/4$	1	0	$-\frac{1}{4}$
-1	2	1	1	1	1	$5/4$	1	0	$-\frac{1}{4}$	0
-1	2	1	1	1	$5/4$	1	0	$-\frac{1}{4}$	0	0
$-\frac{3}{4}$	$\frac{6}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$5/4$	0	$-\frac{1}{4}$	0	0	0
$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	0	0

$h[m,n] =$

	$n=-1$	$n=0$	$n=1$
$m=-1$	-1	3	-1
$m=0$	-2	6	-2
$m=1$	-1	3	-1

$\times \frac{1}{4}$

$$\sum_{m,n} h[m,n] = 1$$

b)

$$H(\mu,\nu) = H_{ver}(\mu)\cdot H_{hor}(\nu)$$

$$H_{ver}(\mu) = \sum_{m=-\infty}^{\infty} h_{ver}(m) e^{-j\mu m} = 1 + \frac{1}{2}e^{-j\mu} + \frac{1}{2}e^{j\mu} = 1 + \cos(\mu)$$

$$H_{hor}(\nu) = \frac{3}{2} - \frac{1}{2}e^{-j\nu} - \frac{1}{2}e^{j\nu} = \frac{3}{2} - \cos(\nu)$$

$$\boxed{H(\mu,\nu) = (1+\cos(\mu))\cdot\left(\frac{3}{2} - \cos(\nu)\right)}$$

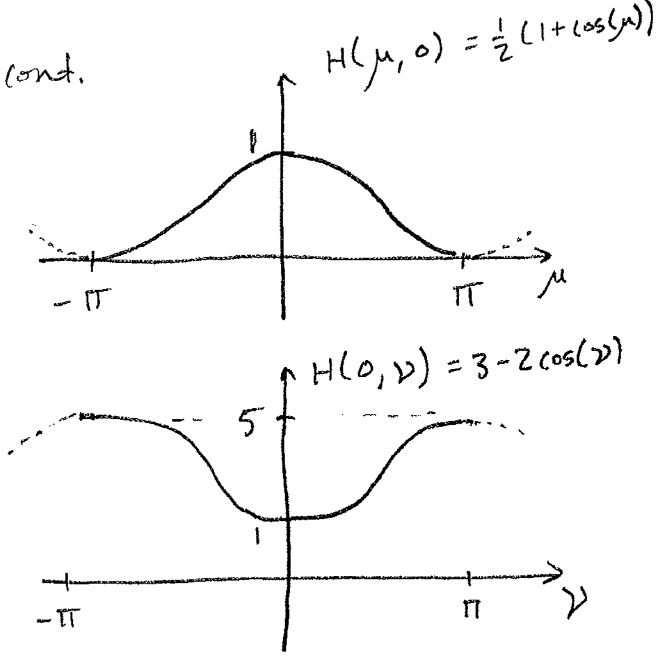
$$H(\mu,0) = \frac{1}{2}(1+\cos(\mu))$$

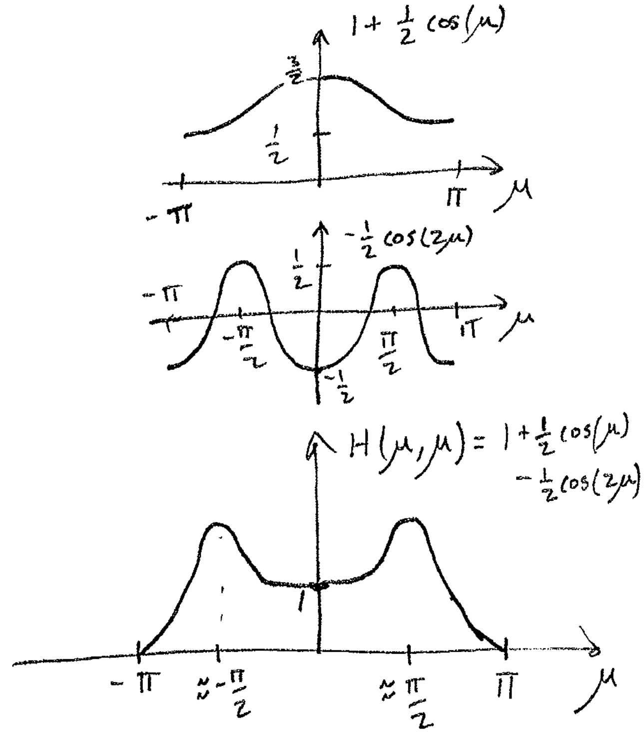
$$H(0,\nu) = 3 - 2\cos(\nu)$$

$$\begin{aligned} H(\mu,\mu) &= \left(\frac{3}{2} - \cos(\mu)\right)(1+\cos(\mu)) \\ &= \frac{3}{2} + \frac{1}{2}\cos(\mu) - \cos^2(\mu) \\ &= \frac{3}{2} + \frac{1}{2}\cos(\mu) - \frac{1+\cos(2\mu)}{2} \\ &= 1 + \frac{1}{2}\cos(\mu) - \frac{1}{2}\cos(2\mu) \end{aligned}$$

5. (continued – 2)

b) cond.





c) Since $H(0,0)=1$, the DC response is 1 which is why the center region of the output is unchanged. (as well as the uniform 0 regions in the corners)

The $H(\mu,0)$ plot shows a low-pass response in the vertical direction which is why horizontal edges are smoothed.

The $H(0,\nu)$ plot shows a high-pass response in the horizontal direction which is why the vertical edge is sharpened.

$H(\mu,\mu)$ characterizes the response in the diagonal direction. The filter cuts off the highest frequencies which is why the diagonal edge transitions have been broadened, but mid-range frequencies have been amplified so we see an over shoot in these edges.