21 March 2012 **Name:** SOLUTION

**ECE 438** **Exam No. 2** **Spring 2012**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider a causal LTI system with transfer function

$$H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}.$$

a. (2) Sketch the locations of the poles and zeros.

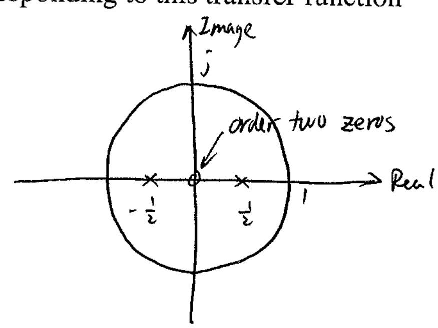
b. (8) Use the graphical approach to determine the magnitude and phase of the frequency response $H(\omega)$, for $\omega = 0$ and $\omega = \pi/2$ radians/sample. (Be sure to show your work.)

c. (2) Is the system stable? Explain why or why not.

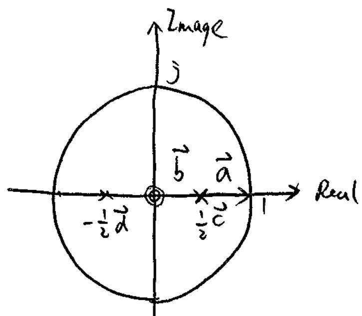
d. (4) Find a difference equation for $y[n]$ in terms of $x[n]$, corresponding to this transfer function $H(z)$.

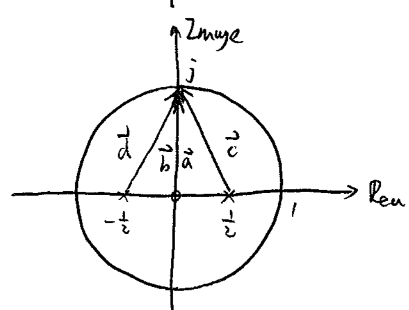
e. (9) Find the impulse response $h[n]$, corresponding to this transfer function $H(z)$.

a. $H(z) = \dfrac{z^2}{(z-\frac{1}{2})(z+\frac{1}{2})}$




b.




$|H(\omega)| = \dfrac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|} = \dfrac{1\times 1}{\frac{1}{2}\times\frac{3}{2}} = \dfrac{4}{3}$

$\angle H(\omega) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d} = 0$

$|H(\omega)| = \dfrac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|} = \dfrac{1\times 1}{\sqrt{1+\frac{1}{4}}\cdot\sqrt{1+\frac{1}{4}}} = \dfrac{4}{5}$

$\angle H(\omega) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d} = \frac{\pi}{2} + \frac{\pi}{2} - \angle\vec{c} - (\pi - \angle\vec{c}) = 0$

1. (continued - 1)

c. the system is stable

Because in causal system, the ROC can be written as $|z| > r$ where $r$ is the largest magnitude of the poles. in this question, $r = \frac{1}{2}$

Therefore, the ROC contains the unit circle, resulting a stable system

d. $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{4}z^{-1}}$

$Y(z) - \frac{1}{4}z^{-2}Y(z) = X(z)$

Compute z-inverse transform and using the time-shifting property

$y[n] - \frac{1}{4}y[n-2] = x[n]$

$y[n] = x[n] + \frac{1}{4}y[n-2]$

e. Expand $H(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}}$

where $A = (1-\frac{1}{2}z^{-1})H(z)\Big|_{z=\frac{1}{2}} = \frac{1}{1+\frac{1}{2}z^{-1}}\Big|_{z=\frac{1}{2}} = \frac{1}{2}$

$B = (1+\frac{1}{2}z^{-1})H(z)\Big|_{z=-\frac{1}{2}} = \frac{1}{1-\frac{1}{2}z^{-1}}\Big|_{z=-\frac{1}{2}} = \frac{1}{2}$

therefore $H(z) = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{2}\cdot\frac{1}{1+\frac{1}{2}z^{-1}}$

Given the ROC $|z| > \frac{1}{2}$ from c.

Using transformation pair

we have $\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}z^{-1}} \xleftrightarrow{z} \frac{1}{2}\cdot(\frac{1}{2})^n u[n]$

$\frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}z^{-1}} \xleftrightarrow{z} \frac{1}{2}\cdot(-\frac{1}{2})^n u[n]$

Using the linearity we have

$h[n] = [\frac{1}{2}\cdot(\frac{1}{2})^n + \frac{1}{2}\cdot(-\frac{1}{2})^n]u[n]$

$= \begin{cases} (\frac{1}{2})^n u[n], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$

2. (25) Consider the two 6-point signals

$$x[n]=\begin{cases} n, & 0<n<6 \\ 0, & \text{else} \end{cases} \quad \text{and} \quad y[n]=\begin{cases} 1, & 0 \le n \le 2 \\ -1, & 3 \le n \le 5 \\ 0, & \text{else} \end{cases}.$$

a. (10) Compute the aperiodic convolution $z[n]$ of these two signals.

b. (13) Compute the 6-point periodic (circular) convolution $w[n]$ of these two signals.

c. (2) Determine to what length $N$ both signals must be padded with zeros for the entire non-zero part of the aperiodic convolution $z[n]$ to be contained within the $N$-point periodic (circular) convolution $w[n]$ for some range of values of $n$.

$n = 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$

$x[n] = 0, 1, 2, 3, 4, 5, 0, 0, 0, \ldots$

$y[n] = 1, 1, 1, -1, -1, -1, 0, 0, 0, \ldots$

a)

$z[0] = 0$
$z[1] = 1$
$z[2] = 1+2=3$
$z[3] = 1+2+3=6$
$z[4] = -1+2+3+4=8$
$z[5] = -1-2+3+4+5=9$
$z[6] = -1-2-3+4+5=3$
$z[7] = -2-3-4+5=-4$
$z[8] = -3-4-5=-12$
$z[9] = -4-5=-9$
$z[10] = -5=-5$

0 1 2 3 4 5 0 0 0 0 . . .
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1
-1 -1 -1 1 1 1

$z[n] = 0 \ \forall \ n \leq 0$
$n \geq 11$

b)

$z[0] = -1-2-3+4+5=3$
$z[1] = 1-2-3-4+5=-3$
$z[2] = 1+2-3-4-5=-9$
$z[3] = 1+2+3-4-5=-3$
$z[4] = -1+2+3+4-5=3$
$z[5] = -1-2+3+4+5=9$
$z[6] = z[0]$

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | -1 | -1 | -1 | 1 | 1 |
| 1 | 1 | -1 | -1 | -1 | 1 |
| 1 | 1 | 1 | -1 | -1 | -1 |
| -1 | 1 | 1 | 1 | -1 | -1 |
| -1 | -1 | 1 | 1 | 1 | -1 |
| -1 | -1 | -1 | 1 | 1 | 1 |
| 1 | -1 | -1 | -1 | 1 | 1 |

$$z[n] = z[n \bmod 6]$$

c) We must pad Both Signals to $N=10$ for the entire non-zero part of the aperiodic convolution to be contained within the N-Point periodic convolution.

$x[n]$ has non zero length $N_x = 5$

$y[n]$ has nonzero length $N_y = 6$

$$N_x + N_y - 1 = 10$$

3. (25 pts) Fast Fourier Transform Algorithm

a) (15) Derive the complete equations that describe a Fast Fourier Transform (FFT) Algorithm to compute the 55-point Discrete Fourier Transform (DFT). Do not provide a flow diagram.

b) (3) Determine the number of complex operations to compute a 55-point DFT by direct evaluation of the expression for the DFT.

c) (4) Determine the number of complex operations to compute a 55 point DFT by using your FFT algorithm derived in part (a) above.

Let $x[n]$ be a 55-point signal. Suppose we zero-pad it to generate a 64-point signal $y[n]$:

$$y[n] = \begin{cases} x[n], & n = 0,1,\ldots,54, \\ 0, & n = 55,36,\ldots,63 \end{cases}$$

d) (3) Determine how many complex operations would be required to compute the 64-point DFT of $y[n]$ using a radix-2 FFT algorithm. Note: you do not need to take into account the fact that some of the data points in the signal $y[n]$ have value zero.

a) Split into 5 11-point DFTs. (Could also do 11 5point DFTs).

$$X^{(55)}[k] = \sum_{n=0}^{54} x[n] e^{-j\frac{2\pi kn}{55}}$$

$$= \sum_{\ell=0}^{4} \sum_{n=0}^{10} x[5n+\ell] e^{-j\frac{2\pi k(5n+\ell)}{55}}$$

$$= \sum_{\ell=0}^{4} e^{-j\frac{2\pi k\ell}{55}} \sum_{n=0}^{10} x[5n+\ell] e^{-j\frac{2\pi kn}{11}}$$

$$= \sum_{\ell=0}^{4} W_{55}^{\ell k} X_{\ell}^{(11)}[k], \quad W_{55}^{\ell k} = e^{-j\frac{2\pi k\ell}{55}}, \quad X_{\ell}^{(11)}[k] = \sum_{n=0}^{10} x[5n+\ell] e^{-j\frac{2\pi kn}{11}}$$

the 11-point DFT of a portion of $x[n]$

b) $N^2 = 55^2 = 3025$

→ Easiest to see by looking at the DFT matrix times the input sequence

c) Five 11-point DFTs: $5(11^2)$

Four complex operations to combine: $4(55)$

$5(11^2) + 4(55) = 605 + 220 = 825$

d) $N \log_2 N = 64 \log_2 64 = 64 \cdot 6 = 384$

4. (25) Let $x[n], n=0,\ldots,N-1$ denote an $N$-point signal, where $N$ is assumed to be even. Let $X^{(N)}[k], k=0,\ldots,N-1$ denote the $N$-point Discrete Fourier Transform (DFT) of $x[n]$. Define an $(N/2)$-point signal $y[n], n=0,\ldots,N/2-1$ according to $y[n]=x[2n], n=0,\ldots,N/2-1$.

Find a simple expression for the $(N/2)$-point DFT $Y^{(N/2)}[k], k=0,\ldots,N/2-1$ in terms of the $N$-point DFT $X^{(N)}[k], k=0,\ldots,N-1$.

FOUR

I'll show two approaches, the first using the knowledge of the DTFT relationship for decimation, and the second sticking strictly to the DFT equations.

Approach 1:

- Assume $x(n)$ non-zero only in the range $0 \le n \le N-1$.
- $y(n) = x[2n]$
- Then the DFT's are related to the DTFT's by:

$$\underbrace{X^{(N)}[k]}_{\text{DFT}} = \underbrace{X(\omega)}_{\text{DTFT}}\Big|_{\omega = 2\pi k/N} \quad (*)$$

$$Y^{(N/2)}[k] = Y(\omega)\Big|_{\omega = 2\pi k/(\frac{N}{2})} \quad (**)$$

- Now from the decimation property,

$$Y(\omega) = \frac{1}{2}\sum_{m=0}^{1} X\left(\frac{\omega - 2\pi m}{2}\right) = \frac{1}{2}X\left(\frac{\omega}{2}\right) + \frac{1}{2}X\left(\frac{\omega-2\pi}{2}\right)$$

So, $$\underbrace{Y^{(\frac{N}{2})}[k] = Y\left(\frac{2\pi k}{(\frac{N}{2})}\right)}_{\text{from } (**)} = \frac{1}{2}X\left(\frac{2\pi k}{N}\right) + \frac{1}{2}X\left(\frac{2\pi k}{N} - \pi\right)$$

$$= \frac{1}{2}X\left(\frac{2\pi k}{N}\right) + \frac{1}{2}X\left(\frac{2\pi}{N}\left(k - \frac{N}{2}\right)\right)$$

from $(*)$

$$= \boxed{\frac{1}{2}X^{(N)}[k] + \frac{1}{2}X^{(N)}\left[k - \frac{N}{2}\right]}$$

or + (periodic w/ period $N$)

4. (continued - 1)

Approach 2 :

$$Y^{(\frac{N}{2})}[k] = \sum_{n=0}^{\frac{N}{2}-1} y(n)\, e^{-j2\pi kn/(\frac{N}{2})} = \sum_{n=0}^{\frac{N}{2}-1} x[2n]\, e^{-j2\pi k 2n/N}$$

(let $l = 2n$) Note this condition

$$= \sum_{\substack{l=0 \\ (l \text{ even})}}^{N-1} x[l]\, e^{-j2\pi kl/N}$$

← need to kill off odd samples to make this look like a DFT.

Define $\frac{1}{2}(1+(-1)^l) = \{1, 0, 1, 0, \dots\}$ with $\uparrow_{l=0}$ under the first 1

$$= \sum_{l=0}^{N-1} x[l]\, \tfrac{1}{2}(1+(-1)^l)\, e^{-j2\pi kl/N}$$

$$= \frac{1}{2}\sum_{l=0}^{N-1} x[l]\, e^{-j2\pi kl/N} + \frac{1}{2}\sum_{l=0}^{N-1} x[l]\, e^{-j\pi l}\, e^{-j2\pi kl/N}$$

$$= \quad || \quad + \frac{1}{2}\sum_{l=0}^{N-1} x[l]\, e^{-j2\pi l(k+\frac{N}{2})/N}$$

$$= \boxed{\frac{1}{2} X^{(N)}[k] + \frac{1}{2} X^{(N)}[k+\tfrac{N}{2}]}$$

4. (25) Let $x[n], n=0,\ldots,N-1$ denote an $N$-point signal, where $N$ is assumed to be even. Let $X^{(N)}[k], k=0,\ldots,N-1$ denote the $N$-point Discrete Fourier Transform (DFT) of $x[n]$. Define an $(N/2)$-point signal $y[n], n=0,\ldots,N/2-1$ according to $y[n]=x[2n], n=0,\ldots,N/2-1$.

Find a simple expression for the $(N/2)$-point DFT $Y^{(N/2)}[k], k=0,\ldots,N/2-1$ in terms of the $N$-point DFT $X^{(N)}[k], k=0,\ldots,N-1$.

APPROACH 3

Yet another solution

$$X^{(N)}[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} = \sum_{m=0}^{\frac{N}{2}-1} x[2m] e^{-j\frac{2\pi k \cdot 2m}{N}} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1] e^{-j\frac{2\pi k(2m+1)}{N}}$$

$$= \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x[2m] e^{-j\frac{2\pi k\cdot m}{\frac{N}{2}}}}_{Y^{(\frac{N}{2})}[k]} + e^{-j\frac{2\pi k}{N}} \cdot \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x[2m+1] e^{-j\frac{2\pi k \cdot m}{\frac{N}{2}}}}_{Z^{(\frac{N}{2})}[k]}, \quad k=0,1,\cdots,N-1$$

Noticing that $Y^{(\frac{N}{2})}[k] = Y^{(\frac{N}{2})}[k+\frac{N}{2}]$, $Z^{(\frac{N}{2})}[k] = Z^{(\frac{N}{2})}[k+\frac{N}{2}]$

$$e^{-j\frac{2\pi k}{N}} = -e^{-j\frac{2\pi(k+\frac{N}{2})}{N}}$$

Therefore, for $k=0,1,\cdots,\frac{N}{2}-1$,

$$X^{(N)}[k] = Y^{(\frac{N}{2})}[k] + e^{-j\frac{2\pi k}{N}} Z^{(\frac{N}{2})}[k] \quad (1)$$

$$X^{(N)}[k+\tfrac{N}{2}] = Y^{(\frac{N}{2})}[k+\tfrac{N}{2}] + e^{-j\frac{2\pi(k+\frac{N}{2})}{N}} Z^{(\frac{N}{2})}[k+\tfrac{N}{2}]$$

$$= Y^{(\frac{N}{2})}[k] - e^{-j\frac{2\pi k}{N}} Z^{(\frac{N}{2})}[k] \quad (2)$$

$$\frac{(1)+(2)}{2} \Rightarrow Y^{(\frac{N}{2})}[k] = \frac{1}{2}\left[X^{(N)}[k] + X^{(N)}[k+\tfrac{N}{2}]\right], \quad k=0,1,\cdots,\frac{N}{2}-1$$

APPROACH 4

$$Y^{(N/2)}[k] = \sum_{n=0}^{N/2-1} x[2n]\, e^{-j2\pi kn/(N/2)}$$

$$x[2n] = \frac{1}{N}\sum_{l=0}^{N-1} X^{(N)}[l]\, e^{j2\pi l 2n/N}$$

$$Y^{(N/2)}[k] = \sum_{n=0}^{N/2-1} \frac{1}{N}\sum_{l=0}^{N-1} X^{(N)}[l]\, e^{+j2\pi l 2n/N}\, e^{-j2\pi kn/(N/2)}$$

$$= \sum_{l=0}^{N-1} X^{(N)}[l]\, \frac{1}{2}\, \frac{1}{N/2}\sum_{n=0}^{N/2-1} e^{j2\pi (l-k)n/(N/2)}$$

$$= \frac{1}{2}\left[X^{(N)}[k] + X^{(N)}[k+N/2]\right]$$