20 April 2011 **Name:** SOLUTION

ECE 438 **Exam No. 3** **Spring 2011**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- Be sure to budget your time so that you can put something down for each problem.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.
- **Note:** There are some new trigonometric identities at the end of the formula sheet that may be helpful for working some of the problems on this exam.

1. (25 pts.) Consider a DT random signal $X[n]$ defined as follows:

$$X[n] = \cos\left(\frac{\pi}{4}n + \Theta\right),$$

where Θ is a random variable uniformly distributed on the interval $[0, 2\pi)$. Each realization or sample function of the random signal $X[n]$ is created by generating a new value for the random variable Θ.

a. (6) Carefully sketch the three realizations or sample functions $x_i[n], i = 1,2,3$ that result for the sample outcomes of the random variable Θ: $\theta_1 = 0$, $\theta_2 = \pi/8$, and $\theta_3 = \pi/2$.

b. (8) Find the expected value $\bar{X}[n] = E\{X[n]\}$ for the random process. **Hint:** You need to take the expectation of an appropriately defined function $g(\Theta, n)$ with respect to the random variable Θ, i.e. you need to compute

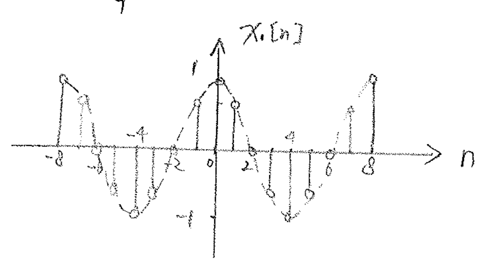
$$\bar{X}[n] = \int g(\theta, n) f_\Theta(\theta) d\theta,$$

where $f_\Theta(\theta)$ is the density function for the random variable Θ.

c. (9) Find the autocorrelation function $R_{XX}[m,n] = E\{X[m]X[n]\}$ for this random process. The hint for part (b) applies here, as well.

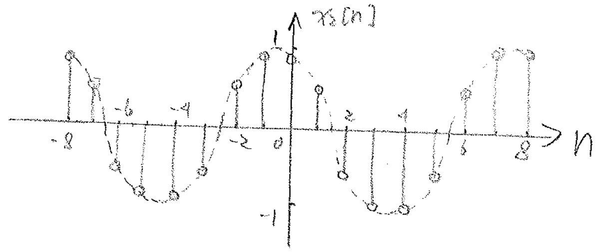
d. (2) Is the random process wide-sense stationary? Explain why or why not.

a. $\theta_1 = 0$ $\quad X_1[n] = \cos\frac{\pi}{4}n$

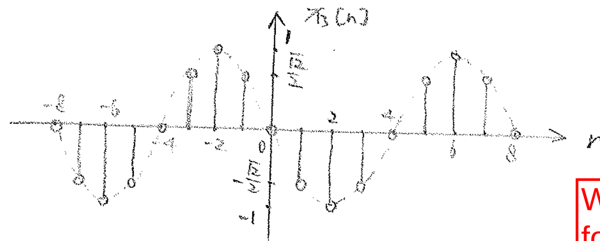


1. (continued)

$\theta_2 = \frac{\pi}{8}$ $x_2[n] = \cos\left(\frac{\pi}{4}n + \frac{\pi}{8}\right)$



$\theta_3 = \frac{\pi}{2}$ $x_3[n] = \cos\left(\frac{\pi}{4}n + \frac{\pi}{2}\right)$



We can just look at this, and see that for any fixed n, we are just integrating over an entire period of cosine, which results in zero. There is no need to formally evaluate the integral.

b. Define $g(\theta, n) = \cos(\theta + \frac{\pi}{4}n)$

Since Θ is uniformly distributed on $[0, 2\pi)$

therefore $f_\Theta(\theta) = \frac{1}{2\pi}$, $0 \le \theta < 2\pi$. where f is the pdf of Θ

$$\bar{X}[n] = \int g(\theta, n) f_\Theta(\theta)\, d\theta = \int_0^{2\pi} \cos(\theta + \tfrac{\pi}{4}n) \frac{1}{2\pi}\, d\theta = \frac{1}{2\pi} \sin(\theta + \tfrac{\pi}{4}n) \Big|_0^{2\pi} = 0$$

c. ~~Be~~ Follow the definition in b.

$$R_{xx}[m,n] = \int g(\theta, m)\, g(\theta, n) f_\Theta(\theta)\, d\theta = \int_0^{2\pi} \cos(\theta + \tfrac{\pi}{4}m) \cos(\theta + \tfrac{\pi}{4}n) \frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \left[\cos(\theta + \tfrac{\pi}{4}m + \theta + \tfrac{\pi}{4}n) + \cos(\theta + \tfrac{\pi}{4}m - \theta - \tfrac{\pi}{4}n) \right] d\theta$$

$$= \frac{1}{8\pi} \sin(2\theta + \tfrac{\pi}{4}m + \tfrac{\pi}{4}n) \Big|_0^{2\pi} + \frac{1}{4\pi} \cos(\tfrac{\pi}{4}m - \tfrac{\pi}{4}n) \Big|_0^{2\pi}$$

$$= \frac{1}{2} \cos\left(\tfrac{\pi}{4}(m-n)\right)$$

d. Yes. The autocorrelation only depends on the difference between start and end point

and the mean is constant!

2. (25 pts.) Consider a 4-level quantizer defined for the signal range $-1 \le x(t) \le 1$. Suppose that the output levels for the quantizer are $y_0 = -0.75$, $y_1 = -0.25$, $y_2 = 0.25$, $y_3 = 0.75$.

   a. (6) Carefully sketch the quantizer function $y = Q(x)$ that for each input x determines the quantized output y.

   Consider the specific CT signal $x(t) = \cos\left(2\pi(10)t\right)$ as the input to the quantizer.

   b. (6) Carefully sketch the quantizer output signal $y(t) = Q\left(x(t)\right)$.

   c. (5) Compute an estimate $\hat{e}_{ms}$ of the mean-squared quantization error based on the assumption that the error is uniformly distributed over its full range.

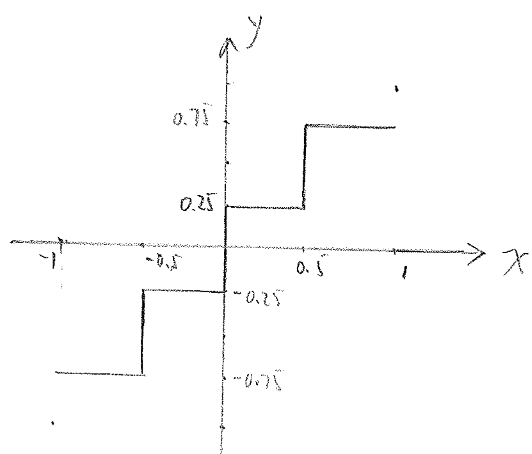
   Now consider that the CT signal is modeled as a random variable X with probability density function given by

$$f_X(x) = \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

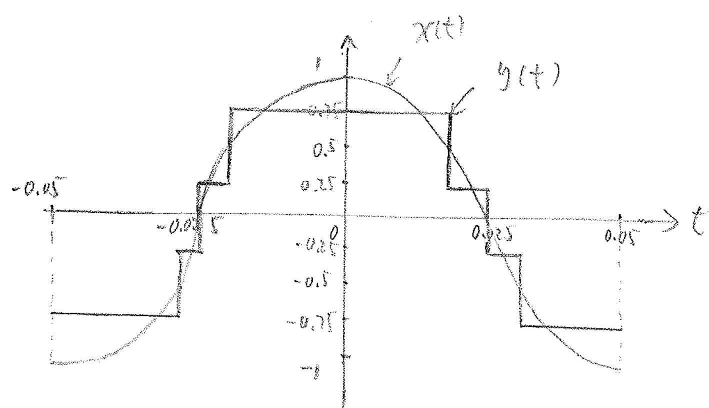
   d. (8) Determine the exact mean-squared quantization error

$$e_{ms} = E\left\{|Y - X|^2\right\} = E\left\{|Q(X) - X|^2\right\}.$$

a.



b.



2. (continued)

c. let $e(t)$ be the quantization error over the range $e(t) \in [-0.25, 0.25]$

Assuming uniformity distribution, we have pdf. $P_e(e) = 2$

$$\hat{e}_{sm} = E[e^2(t)] = \int e^2 P_e(e)\,de$$

$$= \int_{-0.25}^{0.25} e^2 \cdot 2\,de = \frac{2}{3}e^3 \Big|_{-0.25}^{0.25} = \frac{1}{48}$$

or you could just state that it is DELTA^2/12, where DELTA is quantizer step-size.

d. $$e_{ms} = E\{|Q(x) - x|^2\} = \int_0^1 |Q(x) - x|^2 f_x(x)\,dx$$

$$= \int_0^{0.5} (0.25 - x)^2\, 2(1-x)\,dx + \int_{0.5}^1 (0.75 - x)^2\, 2(1-x)\,dx$$

$$= \int_0^{0.5} \left(-2x^3 + 3x^2 - \frac{9}{8}x + \frac{1}{8}\right)dx + \int_{0.5}^1 \left(-2x^3 + 5x^2 - \frac{33}{8}x + \frac{9}{8}\right)dx$$

$$= -\frac{1}{2}x^4 + x^3 - \frac{9}{16}x^2 + \frac{1}{8}x \Big|_0^{0.5} + \left(-\frac{1}{2}x^4 + \frac{5}{3}x^3 - \frac{33}{16}x^2 + \frac{9}{8}x\right)\Big|_{0.5}^1$$

$$= \frac{1}{64} + \frac{35}{64} - \frac{13}{24}$$

$$= \frac{1}{48}$$

3. (25) Consider the DT signal $x[n]$ defined for non-negative time $n \geq 0$ according to:

$$x[n] = \begin{cases} \sin\left(\frac{2\pi}{5}n\right), & n = 0,1,2,3,4, \quad 10,11,12,13,14, \quad 20,21,22,23,24, \quad \ldots \\ 0, & n = 5,6,7,8,9, \quad 15,16,17,18,19, \quad 25,26,27,28,29, \quad \ldots \end{cases}$$

a. (6) Carefully sketch $x[n]$ for $0 \leq n \leq 29$.

Suppose we consider $x[n]$ as a model for a particular speech utterance. Based on this viewpoint, answer parts (b) through (e) below:

b. (1) Is the utterance voiced or unvoiced?

c. (1) Find the pitch period P' in samples.

d. (2) Carefully sketch the DT vocal tract response $v[n]$.

e. (1) Find the formant frequencies ω_i in radians/sample.

Suppose we compute the STDTFT $X(\omega,n)$ of this waveform according to:

$$X(\omega,n) = \sum_k x[k]w[n-k]e^{-j\omega k},$$

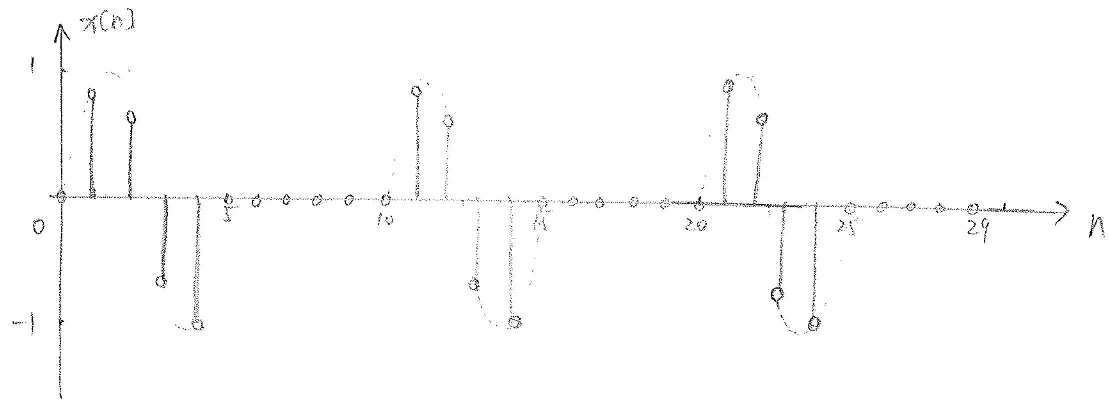
where the window function is defined as $w[k] = \begin{cases} 1, & k = -2,-1,0,1,2 \\ 0, & \text{else} \end{cases}$.

f. (12) Compute $X(\omega,n)$ for $n = 2$. Simplify your answer as much as possible.

g. (1) Compute $X(\omega,n)$ for $n = 7$. Simplify your answer as much as possible.

h. (1) Is this spectrogram wideband or narrowband?

a.



3. (continued)

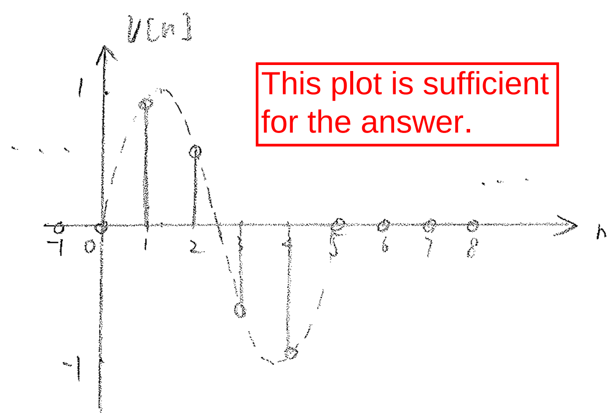
b. the utterance is voiced

c. $p' = 10$ samples

d. $x[n] = v[n] * e[n]$

in this case $e[n] = \sum_{k=-\infty}^{\infty} \delta[n-10k]$

$$v[n] = \begin{cases} \sin\left(\frac{2\pi}{5}n\right), & n = 0, 1, 2, 3, 4 \\ 0, & \text{else} \end{cases}$$



e. $\frac{2\pi}{5}$ radians/sample

3. (continued)

f. when $n=2$

$$w[n-k] = \begin{cases} 1, & k=0,1,2,3,4 \\ 0, & \text{else} \end{cases}$$

$$x[k] = \sin\left(\frac{2\pi}{5}k\right), \quad k=0,1,2,3,4$$

Using DTFT product relation

$$x[n]y[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega-\mu)Y(\mu)d\mu$$

the DTFT transform pair for sine function

$$\sin(\omega_0 n) \overset{DTFT}{\longleftrightarrow} \frac{\pi}{j}\, rep_{2\pi}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$$

and the DTFT shifting property

$$x[n-n_0] \overset{DTFT}{\longleftrightarrow} X(\omega)e^{-j\omega n_0}$$

we get

$$X(\omega,2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\pi}{j}[\delta(\omega-\tfrac{2\pi}{5})-\delta(\omega+\tfrac{2\pi}{5})]\,W(\omega-\mu)e^{-j(\omega-\mu)2}d\mu$$

$$= \frac{1}{2j}W(\omega-\tfrac{2\pi}{5})e^{-j(\omega-\frac{2\pi}{5})\cdot 2} - \frac{1}{2j}W(\omega-\tfrac{2\pi}{5})e^{-j(\omega+\frac{2\pi}{5})2}$$

Since $W(\omega) = psinc_5(\omega)$

Therefore $X(\omega,2) = \frac{1}{2j}psinc_5(\omega-\frac{2\pi}{5})e^{-j(\omega-\frac{2\pi}{5})2} - \frac{1}{2j}psinc_5(\omega+\frac{2\pi}{5})e^{-j(\omega+\frac{2\pi}{5})2}$

g. when $n=7$

$$w[n-k] = \begin{cases} 1, & k=5,6,7,8,9 \\ 0, & \text{else} \end{cases}$$

$$x[k]=0, \quad k=5,6,7,8,9$$

$$X(\omega,7)=0$$

h. the window length is comparable to one pitch period

the spectrogram is wideband

4. (25 pts) Consider the CT signal $x(t) = \frac{1}{t^3}$, defined on the interval $1 \le t \le 2$. We wish to find an optimal approximation $\hat{x}(t) = at + b$ to $x(t)$ over this interval. Here a and b are constants chosen to minimize the total squared error

$$\phi = \int_1^2 \left|\hat{x}(t) - x(t)\right|^2 dt .$$

a) (20) Find the optimal values for a and b.

b) (5) For the values of a and b that you determined in part (a) above, carefully sketch $x(t)$ and $\hat{x}(t)$ on the same axes.

a). Compute ϕ expressed by $a.b$

The solution can be obtained more easily by simply differentiating this integral with respect to a, then separately differentiating it with respect to b.

$$\phi = \int_1^2 \left|at+b - \frac{1}{t^3}\right|^2 dt$$

$$= \int_1^2 \left(a^2t^2 + b^2 + \frac{1}{t^6} + 2abt - \frac{2a}{t^2} - \frac{2b}{t^3}\right) dt$$

$$= \left(\frac{1}{3}a^2t^3 + b^2t - \frac{1}{5}t^{-5} + abt^2 + 2at^{-1} + bt^{-2}\right)\Big|_1^2$$

$$= \frac{7}{3}a^2 + b^2 + \frac{31}{160} + 3ab - a - \frac{3}{4}b$$

this is quadratics with respect to a and b, existing minimum value

let

$$\frac{\partial\phi}{\partial a} = \frac{14}{3}a + 3b - 1 = 0$$

$$\frac{\partial\phi}{\partial b} = 2b + 3a - \frac{3}{4} = 0$$

$$\Longrightarrow \begin{cases} a = -\frac{3}{4} \\ b = \frac{3}{2} \end{cases}$$

b)