23 March 2011 **Name:** SOLUTION

ECE 438 **Exam No. 2** **Spring 2011**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- Be sure to budget your time so that you can put something down for each problem.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the system $y[n] = x[n] - \frac{1}{2}y[n-1]$ with input $x[n] = \left(\frac{1}{3}\right)^n u[n]$.

   Assume that the system is causal, and is initially at rest, i.e. $y[n] = 0, n < 0$.

   a. (3) Find the Z-transform (ZT) $X(z)$ of the input $x[n]$. Be sure to state the region of convergence for $X(z)$.

   b. (4) Find the transfer function $H(z)$ of the system. What is the region of convergence for $H(z)$?

   c. (1) Is this system BIBO stable? Why or why not?

   d. (4) Find the ZT $Y(z)$ of the output. Be sure to state the region of convergence for $Y(z)$.

   e. (13) Use a partial fraction expansion to find the output $y[n]$.

a.
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{3}\right)^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{3}\right)^n z^{-n} = \frac{1}{1-\frac{1}{3}z^{-1}} \quad \left(|z| > \tfrac{1}{3}\right)$$

They can also get full credit by looking this up from the formula sheet.

b. Compute Z transform on both sides of the system

$$Y(z) = X(z) - \tfrac{1}{2}z^{-1}Y(z)$$

$$\Leftrightarrow \left(1 + \tfrac{1}{2}z^{-1}\right)Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+\frac{1}{2}z^{-1}}$$

Since the system is causal, then ROC: $|z| > \frac{1}{2}$

c. ROC contains the unit circle, i.e $\sum_n |h[n]| < \infty$

Therefore the system is BIBO

1. (continued)

"combination" could include intersection or union of these two regions. Better to explicitly state "intersection".

d. $Y(z) = H(z)X(z) = \dfrac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$

the ROC should be the combination of ROC of H & X

therefore ROC of Y is : $|z| > \frac{1}{2}$

e. Solution 1.

Assume $Y(z) = \dfrac{A}{1+\frac{1}{2}z^{-1}} + \dfrac{B}{1-\frac{1}{3}z^{-1}}$, where A, B are to be decided

We want to find A, B

Since $Y(z) = \dfrac{A - \frac{1}{3}Az^{-1} + B + \frac{1}{2}Bz^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \dfrac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$

By comparison of the numerator

$$\begin{matrix} A+B=1 \\ -\frac{1}{3}A+\frac{1}{2}B=0 \end{matrix} \Rightarrow \begin{cases} A=\frac{3}{5} \\ B=\frac{2}{5} \end{cases}$$

$Y(z) = \dfrac{\frac{3}{5}}{1+\frac{1}{2}z^{-1}} + \dfrac{\frac{2}{5}}{1-\frac{1}{3}z^{-1}} \quad (|z| > \frac{1}{2})$

$y[n] = Z^{-1}\{Y(z)\} = \frac{3}{5}\cdot(-\frac{1}{2})^n u[n] + \frac{2}{5}(\frac{1}{3})^n u[n]$

ROC_2 = {z: |z| > 1/3

ROC_1 = {z: |z| > 1/2

ROC_Y = ROC_1 \intersect ROC_2

Solution 2.

We can directly compute A, B using

$A = Y(z)\cdot(1+\frac{1}{2}z^{-1})\Big|_{z=-\frac{1}{2}} = \frac{3}{5}$

$B = Y(z)(1-\frac{1}{3}z^{-1})\Big|_{z=\frac{1}{3}} = \frac{2}{5}$

the rest are the same to Solution 1

2. (25 pts.) Fast Fourier Transform (FFT) algorithm

   a. (2) Write down the equation for the 35-point DFT.

   b. (3) Determine the approximate number of complex operations required to compute the direct 35-point DFT. (A complex operation consists of one complex addition and one complex multiplication.)

   c. (15) Derive the full and complete equations for a 35-point FFT algorithm. (Do not provide a flow diagram.)

   d. (5) Determine the approximate number of complex operations required to compute the 35-point DFT, using your 35-point FFT algorithm derived in part c. above.

a. $X_{35}(k) = \sum_{n=0}^{34} x[n] e^{-j\frac{2\pi kn}{35}}$, $k=0,\cdots,34$

b. for each k we need 35 complex operations

therefore we need in total 35^2 complex operations

write out what is 35^2. Otherwise, we cannot meaningfully compare answers to parts b. and d.

c. $35 = 5\times 7$

We can divide the DFT computation into two levels

the first levels we have five groups of 7-pt DFT

then we use those results to compute 35-pt DFT at the second level

$$X_{35}(k) = \sum_{n=0}^{34} x[n] e^{-j\frac{2\pi kn}{35}}$$

$$\overset{n=5m+l}{=} \sum_{l=0}^{4}\sum_{m=0}^{6} x[5m+l] e^{-j\frac{2\pi k(5m+l)}{35}}$$

$$= \sum_{m=0}^{6} x[5m] e^{-j\frac{2\pi km}{7}} + \sum_{m=0}^{6} x[5m+1] e^{-j\frac{2\pi km}{7}}\cdot e^{-j\frac{2\pi k}{35}} + \sum_{m=0}^{6} x[5m+2] e^{-j\frac{2\pi km}{7}}\cdot e^{-j\frac{2\pi k\cdot 2}{35}} +$$

$$\sum_{m=0}^{6} x[5m+3] e^{-j\frac{2\pi km}{7}} e^{-j\frac{2\pi k\cdot 3}{35}} + \sum_{m=0}^{6} x[5m+4] e^{-j\frac{2\pi km}{7}}\cdot e^{-j\frac{2\pi k\cdot 4}{35}}$$

$$= X_0^{(7)}(k) + X_1^{(7)}(k) e^{-j\frac{2\pi k}{35}} + X_2^{(7)}(k) e^{-j\frac{2\pi\cdot 2k}{35}} + X_3^{(7)}(k) e^{-j\frac{2\pi\cdot 3k}{35}} + X_4^{(7)}(k) e^{-j\frac{2\pi\cdot 4k}{35}}$$

Note: We can also make the first levels contain seven groups of 5-pt DFT

then we need modify the derivation according, though similar

2. (continued)

d. for the first level, each ~~group~~ 7-pt DFT needs $7^2 = 49$ Complex operations

in total is $5 \times 49 = 245$ complex operations

for the second level, to compute each 35-pt DFT we need 5 complex operations

in total is $5 \times 35 = 175$ complex operations

two levels in total is 420 complex operations

Note: the alternative algorithm in C note also has the same # of complex operations

3. (25) Consider the two 8-point signals $x[n]=\begin{cases} 6-n, & n=0,1,2,3,4,5 \\ 0, & n=6,7 \end{cases}$ and

$$y[n]=\begin{cases} 1, & n=0,1,2,3,4 \\ 0, & n=5,6,7 \end{cases}.$$

a. (10) Compute the aperiodic convolution $z[n]$ of these two signals.

b. (12) Compute the 8-point periodic (circular) convolution $w[n]$ of these two signals.

c. (3) Determine to what length N both signals must be padded with zeros for the entire non-zero part of the aperiodic convolution $z[n]$ to be contained within the N-point periodic (circular) convolution $w[n]$ for some range of values of n.

a. $z[n] = x[n] * y[n]$

					$n=$ 0	1	2	3	4	5	6	. . .		
					6	5	4	3	2	1				
$z[0]=6$	1	1	1	1	1									
$z[1]=6+5=11$		1	1	1	1	1								
$z[2]=6+5+4=15$			1	1	1	1	1							
$z[3]=6+5+4+3=18$				1	1	1	1	1						
$z[4]=6+5+4+3+2=20$					1	1	1	1	1					
$z[5]=5+4+3+2+1=15$						1	1	1	1	1				
$z[6]=4+3+2+1=10$							1	1	1	1	1			
$z[7]=3+2+1=6$								1	1	1	1	1		
$z[8]=2+1=3$									1	1	1	1	1	
$z[9]=1$										1	1	1	1	1

for other n, $z[n]=0$

b. $w[n] = x[n] \circledast_8 y[n]$

	$n=$ 0	1	2	3	4	5	6	7	8 . . .
	6	5	4	3	2	1	0	0	
$w[0]=6+2+1=9$	1	0	0	0	1	1	1	1	
$w[1]=6+5+1=12$	1	1	0	0	0	1	1	1	
$w[2]=6+5+4=15$	1	1	1	0	0	0	1	1	
$w[3]=6+5+4+3=18$	1	1	1	1	0	0	0	1	
$w[4]=6+5+4+3+2=20$	1	1	1	1	1	0	0	0	
$w[5]=5+4+3+2+1=15$	0	1	1	1	1	1	0	0	
$w[6]=4+3+2+1=10$	0	0	1	1	1	1	1	0	
$w[7]=3+2+1=6$	0	0	0	1	1	1	1	1	

$w[n]=w[n \bmod 8]$

3. (continued)

c. length of $x[n]$ without zeropadding is $N_1 = 6$

length of $y(n)$ without zeropadding is $N_2 = 5$

the length of aperiodic convolution is $N_1+N_2-1 = 10$

in order to match the resalt of aperiodic with periodic convulution

we must pad zeros to the length of $N=10$ for both sequence

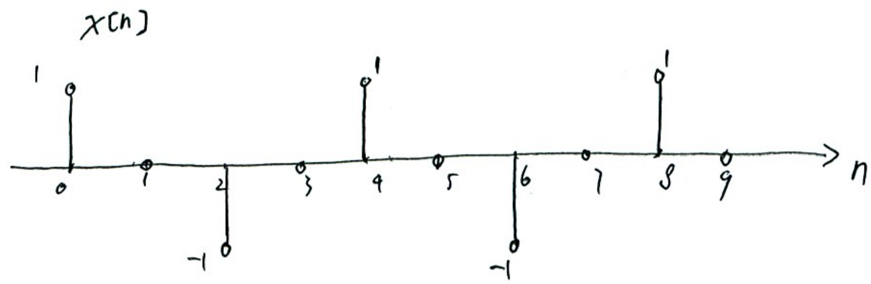
4. (25 pts) Consider the 10-point signal $x[n] = \cos(5\pi n/10), n = 0,...,9$.

   a) (3) Sketch $x[n]$.

   b) (15) Find an expression for the 10-point DFT $X^{10}[k]$ of $x[n]$ in terms of the function $\text{psinc}_N(\omega)=\dfrac{\sin(\omega N/2)}{\sin(\omega/2)}$ for an appropriate value of N.

   c) (7) Approximately sketch $|X^{10}[k]|, k = 0,...,9$.

a.



b. Denote $w[n] = u[n] - u[n-10]$

$y[n] = \cos\left(\frac{5\pi n}{10}\right), \quad n \in \mathbb{Z}$

then $x[n] = y[n]\,w[n]$

Compute DTFT: $W(\omega) = \sum_{n=0}^{9} e^{-j\omega n} = \begin{cases} 10, & \omega = 0, \pm 2\pi \cdots \\ \text{psinc}_{10}(\omega)\cdot e^{-j\omega\frac{9}{2}}, & \text{otherwise} \end{cases}$

$$Y(\omega) = \pi\left[\delta\left(\omega - \frac{5\pi}{10}\right) + \delta\left(\omega + \frac{5\pi}{10}\right)\right]$$

for $|\omega| < \pi$

$$X(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega-\mu)Y(\mu)\,d\mu$$

$$= \cancel{\tfrac{1}{2}\text{psinc}_{10}}\ \frac{1}{2}\left[W\left(\omega - \frac{5\pi}{10}\right) + W\left(\omega + \frac{5\pi}{10}\right)\right]$$

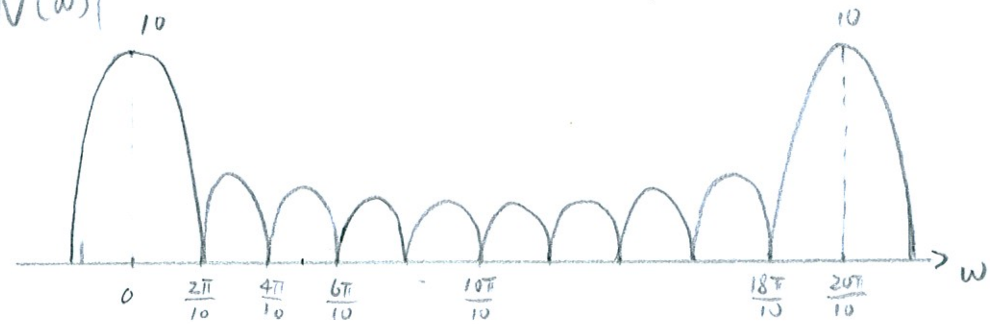
when $N=10$

DFT $X_{10}(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{10}} = \frac{1}{2}\text{psinc}_{10}\left(\frac{\pi}{10}(2k-5)\right)\cdot e^{-j\pi(2k-5)\frac{9}{20}} + \frac{1}{2}\text{psinc}_{10}\left(\frac{\pi}{10}(2k+5)\right)\cdot e^{-j\pi(2k+5)\frac{9}{20}}$

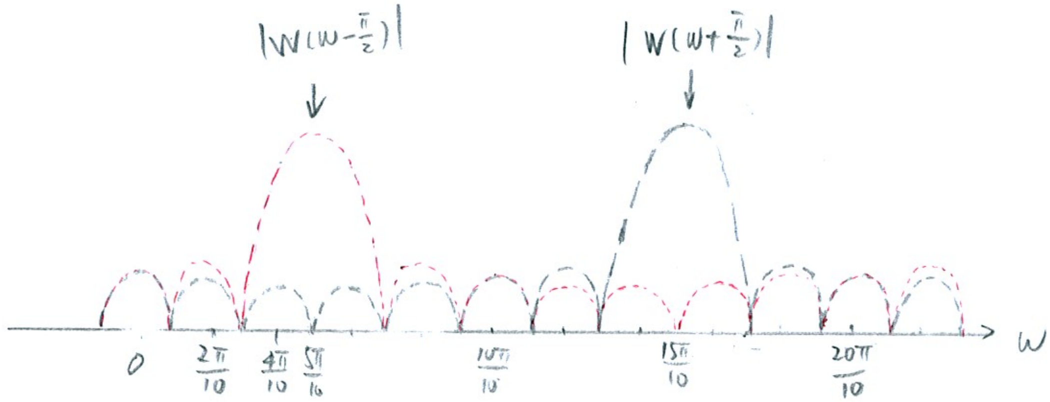
Should probably explicitly state that in part c., you are going to approximate the magnitude of this overall expression by the sum of the magnitudes of the two terms. This approximation is based on the idea that the two terms do not overlap too much.

4. (continued)

c. $|W(\omega)|$



Dash lines add to $|X(\omega)|$



stem $|X_{10}(k)| = \left| X(\omega)\Big|_{\omega=\frac{2\pi k}{10}} \right|$