21 April 2010 **Name:** SOLUTION

ECE 438 **Exam No. 3** **Spring 2010**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider a 3-level quantizer defined as

$$Q(x) = \begin{cases} -1, & -\frac{3}{2} \le x < -\frac{1}{2} \\ 0, & -\frac{1}{2} \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le \frac{3}{2} \end{cases} . \tag{1}$$

a. (8) Suppose that the input to this quantizer is the waveform $x(t) = \sin(2\pi t/12)$ where the units of time t are seconds. Carefully sketch the waveforms $x(t)$ and $y(t) = Q(x(t))$ on the same axes for the interval $0 \le t \le 1$. Be sure to dimension both axes of your plot.

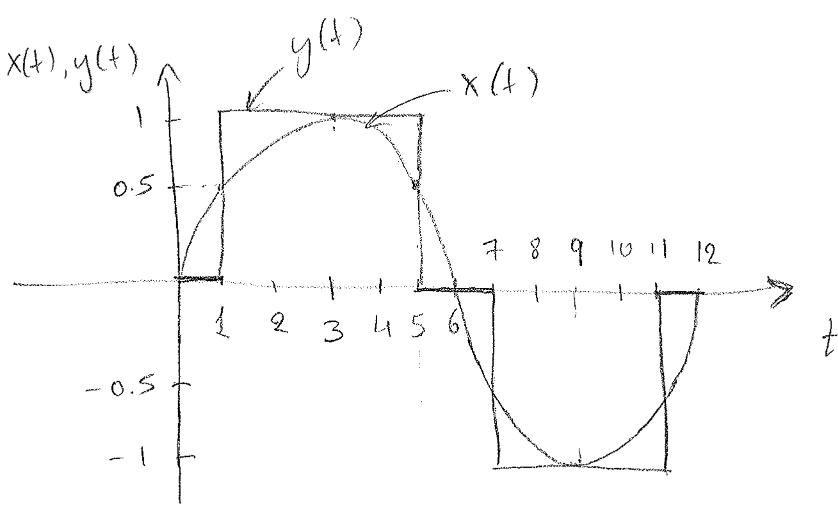
b. (4) Using the formula derived in class that is based on the assumption that the quantizer error is uniformly distributed over the quantizer step-size, calculate the mean-squared quantization error for the quantizer given by Eq. (1) above. For this problem, you should assume that the range of the input signal is from $-\frac{3}{2}$ to $\frac{3}{2}$.

c. (13) Now suppose that the input to the quantizer given by Eq. (1) is a random variable X with first order probability density function

$$f_X(x) = \begin{cases} |x|, & |x| < 1 \\ 0, & \text{else} \end{cases} . \tag{2}$$

Compute the exact mean-squared quantization error $e_{ms} = E\left\{[Q(X) - X]^2\right\}$ for the quantizer given by Eq. (1) and the input with density given by Eq. (2).

(a)



1. (continued)

(b) $\Delta = 1$, The approximate mean squared quantization error is given as $\frac{\Delta^2}{12} = \frac{1}{12}$

(c) $$Y = \begin{cases} -1 & -\frac{3}{2} \le x < -\frac{1}{2} \\ 0 & -\frac{1}{2} \le x < \frac{1}{2} \\ +1 & \frac{1}{2} \le x \le \frac{3}{2} \end{cases}$$

$$(Y-X)^2 = \begin{cases} (-1-x)^2 & -\frac{3}{2} \le x < -\frac{1}{2} \\ (0-x)^2 & -\frac{1}{2} \le x < \frac{1}{2} \\ (1-x)^2 & \frac{1}{2} \le x \le \frac{3}{2} \end{cases}$$

$$e_{ms} = E\{(Y-X)^2\} = \int_{-\frac{3}{2}}^{-\frac{1}{2}} (x+1)^2 f_X(x)\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 f_X(x)\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} (x-1)^2 f_X(x)\,dx$$

$$= -\int_{-1}^{-\frac{1}{2}} (x+1)^2 x\,dx - \int_{-\frac{1}{2}}^{0} x^2 \cdot x\,dx + \int_{0}^{\frac{1}{2}} x^2 \cdot x\,dx + \int_{\frac{1}{2}}^{1} (x-1)^2 x\,dx$$

$$= -\int_{-1}^{-\frac{1}{2}} (x^3+2x^2+x)\,dx - \int_{-\frac{1}{2}}^{0} x^3\,dx + \int_{0}^{\frac{1}{2}} x^3\,dx + \int_{\frac{1}{2}}^{1} (x^3-2x^2+x)\,dx$$

$$= -\left(\frac{x^4}{4} + \frac{2}{3}x^3 + \frac{x^2}{2}\right)\Bigg|_{-1}^{-\frac{1}{2}} - \frac{x^4}{4}\Bigg|_{-\frac{1}{2}}^{0} + \frac{x^4}{4}\Bigg|_{0}^{\frac{1}{2}} + \left(\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{x^2}{2}\right)\Bigg|_{\frac{1}{2}}^{1}$$

$$= -\left(\frac{1}{64} - \frac{1}{12} + \frac{1}{8} - \frac{1}{4} + \frac{2}{3} - \frac{1}{2}\right) + \frac{1}{64} + \frac{1}{64} + \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - \frac{1}{64} + \frac{1}{12} - \frac{1}{8}$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \frac{4}{3} + 1 = \frac{1}{12}$$

2. (25 pts.) Consider four random variables X_0, X_1, X_2, and X_3 that are independent and identically distributed, each with mean 2 and variance 1. Now suppose that we define three new random variables

$$\begin{aligned} Y_0 &= \tfrac{1}{2}(X_0 + X_1) \\ Y_1 &= \tfrac{1}{2}(X_1 + X_2). \\ Y_2 &= \tfrac{1}{2}(X_2 + X_3) \end{aligned} \qquad (1)$$

a. (3) Compute the expected value (mean) of each of the random variables Y_0, Y_1, and Y_2.

b. (6) Compute the variance of each of the random variables Y_0, Y_1, and Y_2.

c. (8) Compute the correlation $E\{Y_0Y_1\}$, covariance $\sigma^2_{Y_0Y_1} = E\{(Y_0 - \bar{Y}_0)(Y_1 - \bar{Y}_1)\}$, and correlation coefficient $\rho_{Y_0Y_1}$ for the two random variables Y_0 and Y_1.

d. (6) Compute the correlation $E\{Y_0Y_2\}$, covariance $\sigma^2_{Y_0Y_2} = E\{(Y_0 - \bar{Y}_0)(Y_2 - \bar{Y}_2)\}$, and correlation coefficient $\rho_{Y_0Y_2}$ for the two random variables Y_0 and Y_2.

e. (2) Comment on the relationship between your answers to parts (c) and (d) above. Specifically, what does this have to do with filtering of random sequences, and the relation between the autocorrelation function of the input to the filter and the autocorrelation function of the output of the filter?

(a) $E[Y_0] = \frac{1}{2}E[(X_0+X_1)] = \frac{1}{2}E[X_0] + \frac{1}{2}E[X_1] = \frac{2+2}{2} = \underline{\underline{2}}$

$E[Y_1] = \frac{1}{2}E[(X_1+X_2)] = \frac{1}{2}E[X_1] + \frac{1}{2}E[X_2] = \frac{2+2}{2} = \underline{\underline{2}}$

$E[Y_2] = \frac{1}{2}E[(X_2+X_3)] = \frac{1}{2}E[X_2] + \frac{1}{2}E[X_3] = \frac{2+2}{2} = \underline{\underline{2}}$

(b) $E[Y_0^2] = E\left[\frac{1}{4}(X_0^2 + X_1^2 + 2X_0X_1)\right] = \frac{1}{4}\left(E[X_0^2] + E[X_1^2] + 2E[X_0X_1]\right)$

$= \frac{1}{4}\left(E[X_0^2] + E[X_1^2] + 2E[X_0]E[X_1]\right)$ ($\because$ X_0 and X_1 are ind.)

$= \frac{1}{4}(5 + 5 + 8) = \frac{9}{2}$

$Var(Y_0) = E[Y_0^2] - (E[Y_0])^2 = \frac{9}{2} - 4 = \frac{1}{2}$

Using similar arguements we have that

$$E[Y_1^2] = \frac{1}{4}\left(E[X_1^2] + E[X_2^2] + 2E[X_1]E[X_2]\right) = \frac{9}{2}$$

$$\Rightarrow \quad Var(Y_1) = E[Y_1^2] - (E[Y_1])^2 = \frac{1}{2}$$

$$E[Y_2^2] = \frac{1}{4}\left(E[X_2^2] + E[X_3^2] + 2E[X_2]E[X_3]\right) = \frac{9}{2}$$

$$\Rightarrow Var(Y_2) = E[Y_2^2] - (E[Y_2])^2 = \frac{1}{2}$$

c) $$E[Y_0Y_1] = E\left[\frac{1}{2}(X_0+X_1)\frac{1}{2}(X_1+X_2)\right] = \frac{1}{4}E\left[X_0X_1 + X_0X_2 + X_1X_2 + X_1^2\right]$$

$$= \frac{1}{4}\left(E[X_0X_1] + E[X_0X_2] + E[X_1X_2] + E[X_1^2]\right)$$

$$= \frac{1}{4}\left(E[X_0]E[X_1] + E[X_0]E[X_2] + E[X_1]E[X_2] + E[X_1^2]\right) \quad (\because X_0, X_1, X_2 \text{ are ind})$$

$$= \frac{1}{4}(4+4+4+5) = \frac{17}{4}$$

$$\sigma^2_{Y_0Y_1} = E[(Y_0 - \bar{Y_0})(Y_1 - \bar{Y_1})] = E[Y_0Y_1 - Y_0\bar{Y_1} - \bar{Y_0}Y_1 + \bar{Y_0}\bar{Y_1}]$$

$$= E[Y_0Y_1] - E[Y_0\bar{Y_1}] - E[\bar{Y_0}Y_1] + E[\bar{Y_0}\bar{Y_1}]$$

$$= \frac{17}{4} - 4 - 4 + 4 = \frac{1}{4}$$

$$\rho_{Y_0Y_1} = \frac{\sigma^2_{Y_0Y_1}}{\sigma_{Y_0}\sigma_{Y_1}} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{4}}} = \frac{1}{2}$$

d) Using similar arguements as (c) we have

$$E[Y_0Y_2] = \frac{1}{4}\left(E[X_0]E[X_3] + E[X_0]E[X_2] + E[X_1]E[X_3] + E[X_1]E[X_2]\right)$$

$$= \frac{16}{4} = 4$$

2. (continued)

$$\sigma^2_{y_0y_2} = E\left[(y_0 - \bar{y}_0)(y_2 - \bar{y}_2)\right]$$

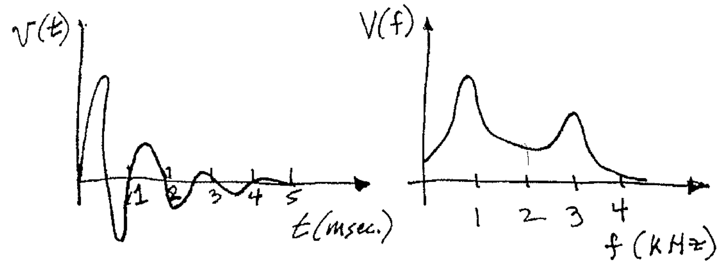
$$= E[y_0 y_2] - E[y_0 \bar{y}_2] - E[\bar{y}_0 y_2] + E[\bar{y}_0 \bar{y}_2]$$

$$= 4 - 4 - 4 + 4 = 0$$

$$\therefore \rho_{y_0y_2} = \frac{\sigma^2_{y_0y_2}}{\sigma_{y_0}\sigma_{y_2}} = 0$$

e) It can be seen that $\rho_{y_0y_1} \neq \rho_{y_0y_2}$ from (c) and (d). Furthermore, we can see that even if the inputs x_0, x_1, x_2, x_3 are independent identically distributed and uncorrelated, the outputs y_0, y_1, y_2 are correlated, even if the system is LTI

3. (25) Consider a voiced phoneme for which the time-domain continuous-time vocal tract response $v(t)$ and corresponding frequency response (CTFT) $V(f)$ are given below.

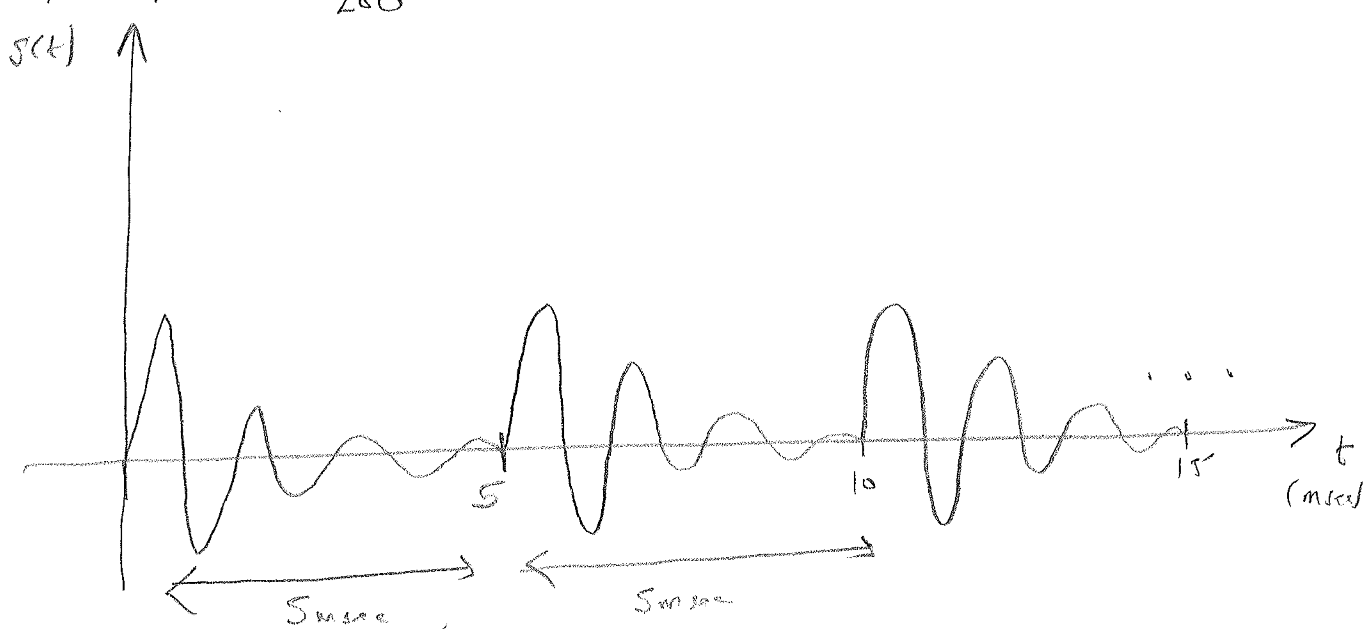


a. (5) Assume that the pitch frequency for the speaker is 200 Hz. Sketch what the continuous-time domain speech waveform $s(t)$ would look like in this case. Be sure to dimension all important quantities in the speech waveform.

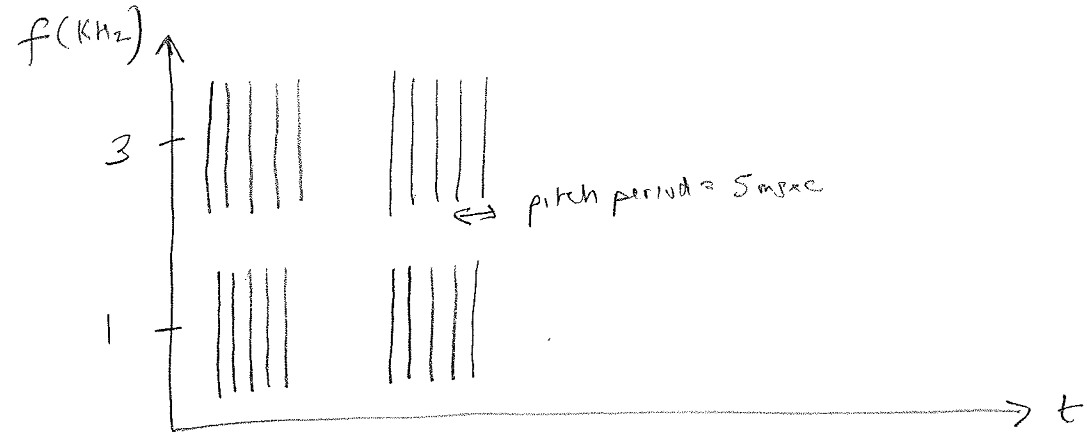
b. (10) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 25 samples. Carefully sketch the resulting spectrogram. Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

c. (10) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 1,000 samples. Carefully sketch the resulting spectrogram. Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

(a) pitch period $= \frac{1}{200} = 5$ msec

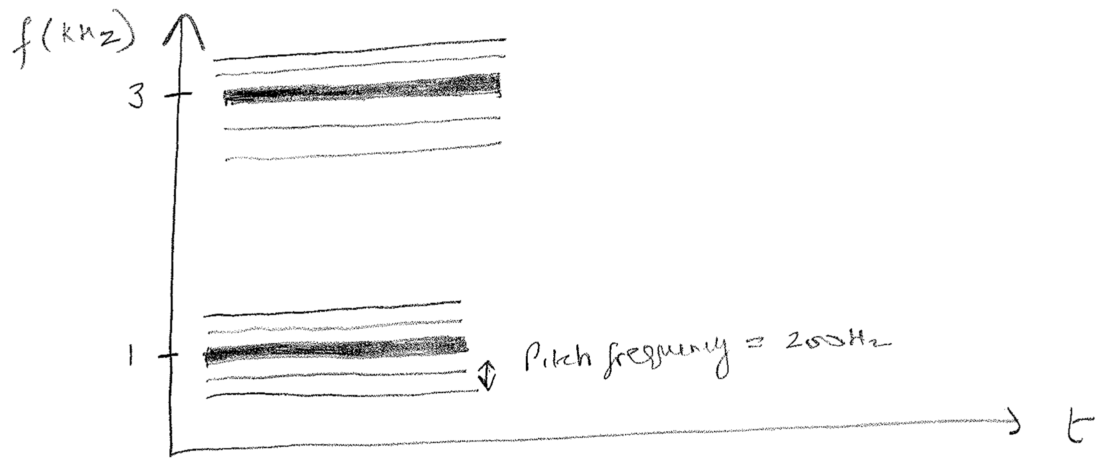


(b) The formant frequencies are at $F_1 = 1\,KHz$ and $F_2 = 3\,KHz$



This is a wideband spectrum because we have a shorter window length which gives good time domain resolution but poor frequency domain resolution

c)



This is a narrowband spectrum because we have a longer time window length which gives good frequency domain resolution but poor time domain resolution

3. (continued)

4. (25 pts) Consider the signal $x(t) = t^2$ that is defined over the interval $0 \le t \le 1$. We wish to approximate $x(t)$ over this interval by $\hat{x}(t) = a_0 s_0(t) + a_1 s_1(t)$, where the basis functions $s_0(t)$ and $s_1(t)$ are given by

$$\begin{aligned} s_0(t) &= 1, \\ s_1(t) &= t, \end{aligned} \quad 0 \le t \le 1, \tag{1}$$

and the terms a_0 and a_1 are constants.

a. (18) Determine the values for a_0 and a_1 that minimize the mean-squared approximation error

$$e_{\mathrm{ms}} = \int_0^1 [\hat{x}(t) - x(t)]^2 dt .$$

b. (7) Using your values for a_0 and a_1 determined in part (a) above, carefully sketch $x(t)$ and $\hat{x}(t)$ on the same axes for the interval $0 \le t \le 1$. Be sure to label and dimension both axes.

(a) $\hat{x}(t) = a_0 + a_1 t \qquad 0 \le t \le 1$

$x(t) = t^2$

$$e_{ms} = \int_0^1 (\hat{x}(t) - x(t))^2 dt = \int_0^1 (a_1 t + a_0 - t^2)^2 dt$$

$$\frac{\partial e_{ms}}{\partial a_0} = 2\int_0^1 (a_1 t + a_0 - t^2)\, dt = 2\left(\frac{a_1}{2} + a_0 - \frac{1}{3}\right)$$

Need to set $\frac{\partial e_{ms}}{\partial a_0} = 0 \Rightarrow 3a_1 + 6a_0 = 2$ — ①

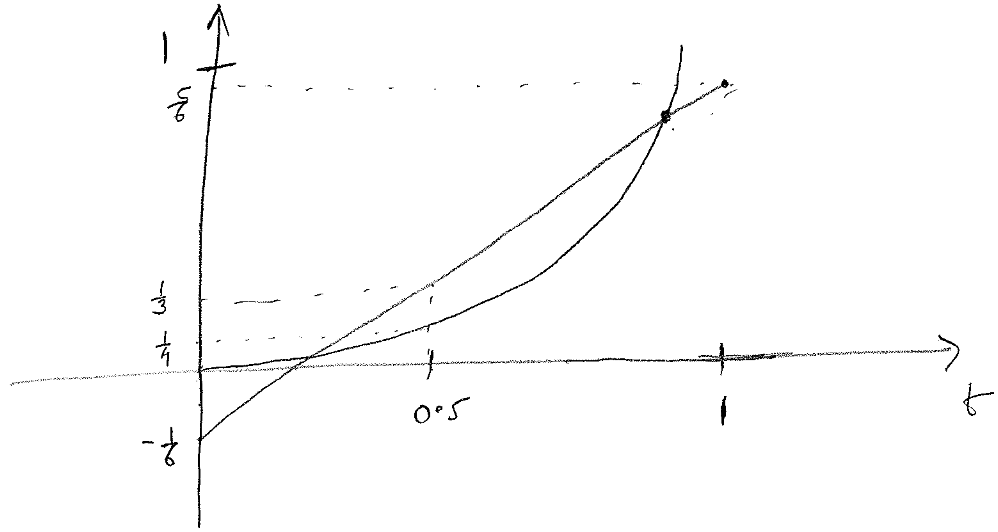
$$\frac{\partial e_{ms}}{\partial a_1} = 2\int_0^1 (a_1 t + a_0 - t^2)\, t\, dt = 2\left(\frac{a_1}{3} + \frac{a_0}{2} - \frac{1}{4}\right)$$

Need to set $\frac{\partial e_{ms}}{\partial a_1} = 0 \Rightarrow 4a_1 + 6a_0 = 3$ — ②

Solving ① and ② simultaneously we get

$a_1 = 1, \quad a_0 = -\frac{1}{6}$

(b)



$x(t) = t^2$

$\hat{x}(t) = t - \frac{1}{6}$

4. (continued)

1.	______	**/25**
2.	______	**/25**
3.	______	**/25**
4.	______	**/25**
Total	______	**/100**