24 March 2010 **Name:** Solution

ECE 438 **Exam No. 2** **Spring 2010**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (28 pts.) Consider a discrete-time, causal, linear, time-invariant system with input $x[n]$, output $y[n]$, and impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

a. (7) Find the transfer function $H(z)$ for this system. Be sure to state the region of convergence.

b. (3) Plot the poles and zeros for this system in the complex z-plane. (Be sure to multiply numerator and denominator by z raised to a sufficiently large power so that you clear all negative powers of z. This is necessary to be sure that you find all the zeros.)

c. (2) Is this system BIBO stable? (Why or why not?)

Suppose that the input to this system is a signal $x[n]$ with Z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| < 1.$$

d. (4) Find the Z-transform $Y(z)$ of the output $y[n]$. Be sure to state the region of convergence.

e. (12) Find the output $y[n]$ by determining the inverse Z-transform of your answer to part d).

Key formula $\quad a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \; |z| > |a|$

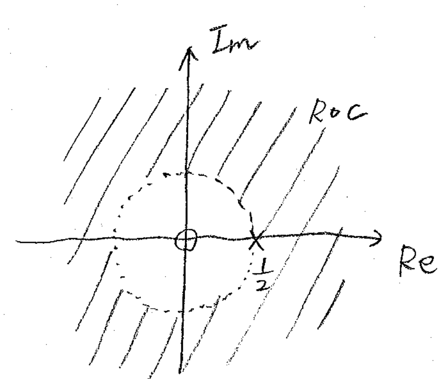
$$-a^n u[-n-1] \leftrightarrow \frac{1}{1 - az^{-1}}, \; |z| < |a|$$

(a) $\quad H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad |z| > \frac{1}{2}$

(b) $\quad H(z) = \frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2}$

Zeros: $z = 0$

poles: $z = \frac{1}{2}$



1. (continued)

(c) This system is BIBO stable because the RoC contains the unit circle.

(d) $X(z) = \dfrac{1}{1 - z^{-1}}, \quad |z| < 1$

RoC of $H(z)$: $|z| > \frac{1}{2}$
RoC of $X(z)$: $|z| < 1$
$\Big\}$ $(|z| > \frac{1}{2}) \cap (|z| < 1) = (\frac{1}{2} < |z| < 1)$

$$Y(z) = H(z) \cdot X(z)$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} \cdot \frac{1}{1 - z^{-1}}$$

$$= \frac{z^2}{(z - \frac{1}{2})(z - 1)}, \quad \frac{1}{2} < |z| < 1$$

(e) $\dfrac{Y(z)}{z} = \dfrac{z}{(z - \frac{1}{2})(z - 1)} = \dfrac{A}{z - \frac{1}{2}} + \dfrac{B}{z - 1}$

$$A = \frac{z}{z-1}\Big|_{z = \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} - 1} = -1$$

$$B = \frac{z}{z - \frac{1}{2}}\Big|_{z = 1} = \frac{1}{\frac{1}{2}} = 2$$

$$\therefore Y(z) = z\left(\frac{-1}{z - \frac{1}{2}} + \frac{2}{z - 1}\right)$$

$$= \frac{-1}{1 - \frac{1}{2} z^{-1}} + \frac{2}{1 - z^{-1}}, \quad \frac{1}{2} < |z| < 1$$

$$\therefore y[n] = -\left(\frac{1}{2}\right)^n u[n] - 2 \cdot 1^n u[-n-1]$$

$$= -\left(\frac{1}{2}\right)^n u[n] - 2u[-n-1]$$

2. (28 pts.) Consider the 8-point signal $x[n] = e^{j\pi n/2}$, $n = 0, ..., 7$.

   a. (3) Carefully plot $x[n]$ in the complex plane (real part vs. imaginary part) for $n = 0, ..., 3$.

   b. (4) Find the 8-point discrete Fourier transform (DFT) $X^{(8)}[k]$, $k = 0, ..., 7$, for this signal. (Use the formula sheets as much as possible to solve this part.)

   c. (2) Sketch its magnitude $\left|X^{(8)}[k]\right|$ for $k = 0, ..., 7$.

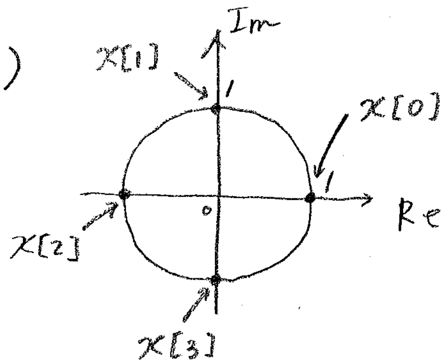
   Now consider a second 8-point signal $y[n] = e^{j5\pi n/8}$, $n = 0, ..., 7$.

   d. (3) Carefully plot $y[n]$ in the complex plane (real part vs. imaginary part) for $n = 0, ..., 3$.

   e. (10) Find the 8-point discrete Fourier transform (DFT) $Y^{(8)}[k]$, $k = 0, ..., 7$, for this signal. (Use the formula sheets as much as possible to solve this part.)

   f. (6) Sketch its magnitude $\left|Y^{(8)}[k]\right|$ for $k = 0, ..., 7$.

(a)

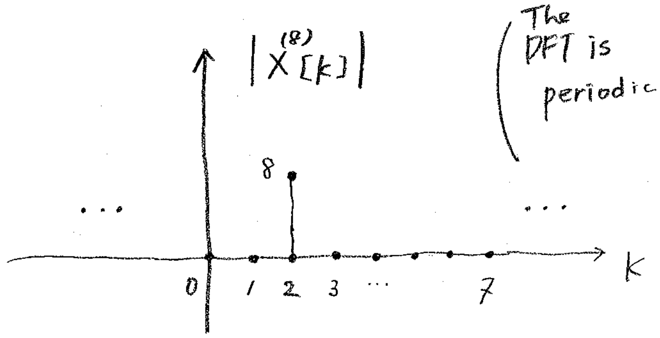


(b) From the formula

$$e^{\frac{j2\pi k_0 n}{N}} \overset{DFT}{\longleftrightarrow} N\,\delta[k-k_0], \quad 0 \le k \le N-1,$$
$$0 \le n \le N-1$$

$$X^{(8)}[k] = 8\,\delta[k-2], \quad 0 \le k \le 7$$

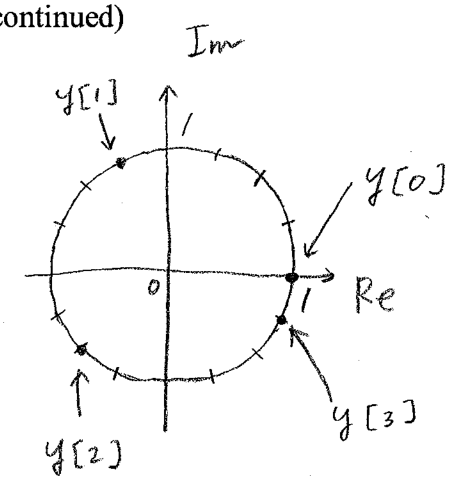
(c)



(The DFT is periodic with 8)

2. (continued)

(d)



(e) From the formula sheet,

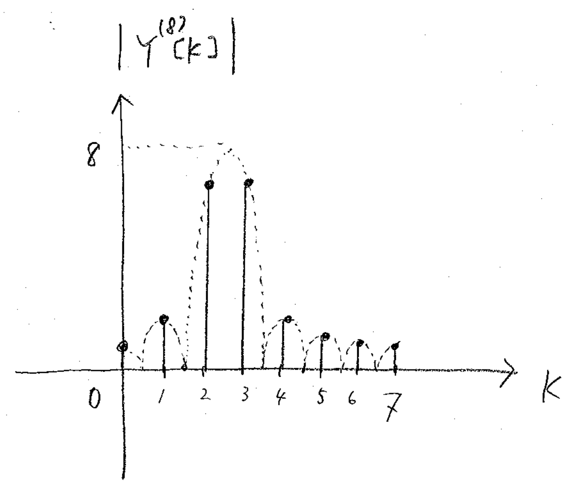
$$e^{j\omega_0 n} \xleftrightarrow{DFT} \frac{\sin\left(\left(\frac{2\pi k}{N}-\omega_0\right)\cdot\frac{N}{2}\right)}{\sin\left(\frac{\frac{2\pi k}{N}-\omega_0}{2}\right)} \cdot e^{j\left(\frac{2\pi k}{N}-\omega_0\right)\cdot\frac{N-1}{2}}$$

$n = 0, \cdots N-1$

$\omega_0 = \frac{5\pi}{8}, \quad N = 8$

$$\therefore \; Y^{(8)}[k] = \frac{\sin\left(\frac{\pi(2k-5)}{2}\right)}{\sin\left(\frac{\pi(2k-5)}{16}\right)} \cdot e^{j\cdot\frac{\pi(2k-5)}{8}\cdot\frac{7}{2}}$$

(f) $$\left|Y^{(8)}[k]\right| = \left|\frac{\sin\left(\frac{\pi(2k-5)}{2}\right)}{\sin\left(\frac{\pi(2k-5)}{16}\right)}\right|$$



3. (15) Consider the following signal $x(t) = 2\cos\big(2\pi(1000)t\big) + 0.5\cos\big(2\pi(6000)t\big)$. Suppose this signal is sampled at a 10 kHz rate to generate 1024 data points $x[n], n = 0, \ldots, 1023$.

Determine the approximate location (in terms of k) and amplitude of each spectral peak in the 1024-point DFT $X^{(1024)}[k], k = 0, \ldots, 1023$, of $x[n], n = 0, \ldots, 1023$.

Receipe

$$\begin{cases} - \ x[n] = x(nT_s) \\ - \ x_{tr}[n] = x[n]\cdot \underbrace{w[n]}_{\uparrow \text{ truncating window}} \\ - \ \text{Find } X_{tr}(\omega) \\ - \ \text{Calculate } X^{(1024)}[k] \text{ using } X^{(N)}[k] = X(\omega)\big|_{\omega = \frac{2\pi k}{N}} \end{cases}$$

$$x[n] = x(nT_s) \qquad \text{where } T_s = \frac{1}{10\cdot 10^3}$$

$$= 2\cos(2\pi\cdot 1000\cdot nT_s) + 0.5\cos(2\pi\cdot 6000\cdot nT_s)$$

$$= 2\cos\left(2\pi\cdot n\cdot \tfrac{1}{10}\right) + 0.5\cos\left(2\pi\cdot 6\cdot n\cdot \tfrac{1}{10}\right)$$

$$= 2\cos\left(\tfrac{\pi}{5}n\right) + 0.5\cos\left(\tfrac{6\pi}{5}\cdot n\right)$$

$$w[n] = u[n] - u[n-1024]$$

$$x_{tr}[n] = x[n]\cdot w[n]$$

Using the formula $\cos(\omega_0 n) \xleftrightarrow{DTFT} \pi\, \text{rep}_{2\pi}\big(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\big)$,

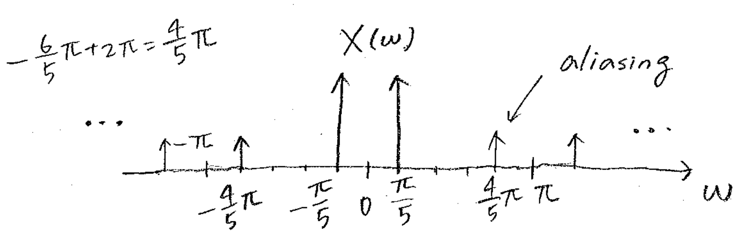
$$X(\omega) = \text{DTFT}\{x[n]\}$$

NOTE: Aliasing from $-\frac{6}{5}\pi + 2\pi = \frac{4}{5}\pi$

$$= 2\pi\left\{\delta\left(\omega - \tfrac{\pi}{5}\right) + \delta\left(\omega + \tfrac{\pi}{5}\right)\right\} + 0.5\pi\left\{\delta\left(\omega - \tfrac{4}{5}\pi\right) + \delta\left(\omega + \tfrac{4}{5}\pi\right)\right\}$$

for $-\pi \le \omega \le \pi$

(periodic with 2π)



3. (continued)

$$X_{tr}(\omega) = \text{DTFT}\{x_{tr}[n]\}$$
$$= \text{DTFT}\{x[n]\cdot w[n]\}$$

periodic convolution

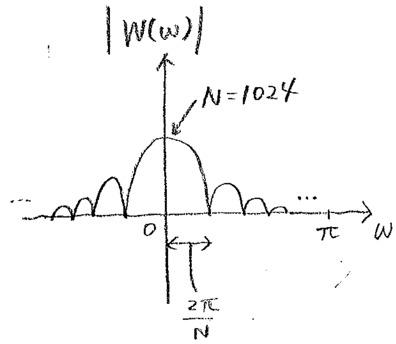
$$= X(\omega) \circledast W(\omega)$$

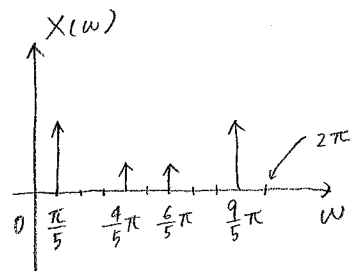
More specifically,

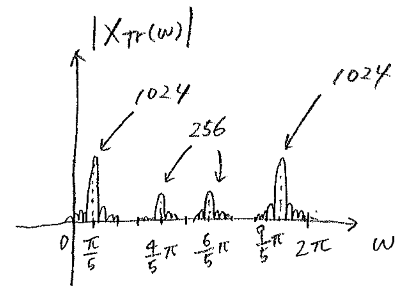
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega-\mu)\, W(\mu)\, d\mu$$
$$= \frac{1}{2\pi}\cdot 2\pi\left(W(\omega-\tfrac{\pi}{5}) + W(\omega+\tfrac{\pi}{5})\right) + \frac{1}{2\pi}\cdot 0.5\pi\left(W(\omega-\tfrac{4}{5}\pi) + W(\omega+\tfrac{4}{5}\pi)\right)$$
$$= W(\omega-\tfrac{\pi}{5}) + W(\omega+\tfrac{\pi}{5}) + \frac{1}{4}\left\{W(\omega-\tfrac{4}{5}\pi) + W(\omega+\tfrac{4}{5}\pi)\right\}$$

where $W(\omega) = \text{DTFT}\{w[n]\}$

$$= \frac{\sin(\frac{1024}{2}\omega)}{\sin(\frac{\omega}{2})}\, e^{-j\omega(\frac{1023}{2})}, \quad \text{for } -\pi \le \omega \le \pi .$$







(NOTE: plotted for $0 \le \omega \le 2\pi$ here to find the corresponding k)

Since $X^{(1024)}[k] = X_{tr}(\omega)\Big|_{\omega = \frac{2\pi k}{1024}}$,

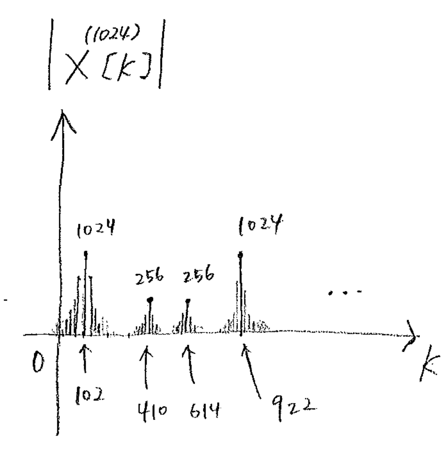
$X^{(1024)}[k]$ has peaks approximately at

Answer

$$k \doteqdot \frac{\pi}{5}\times\frac{1024}{2\pi} = 102.4 \Rightarrow 102$$
$$\frac{4}{5}\pi\times\frac{1024}{2\pi} = 409.6 \Rightarrow 410$$
$$\frac{6}{5}\pi\times\frac{1024}{2\pi} = 614.4 \Rightarrow 614$$
$$\frac{9}{5}\pi\times\frac{1024}{2\pi} = 921.6 \Rightarrow 922$$

These are peaks in range [0, 1023].

$$\left|X^{(1024)}[102]\right| = X_{tr}(\omega)\Big|_{\omega=\frac{2\pi\cdot 102}{1024}} \doteqdot X_{tr}(\tfrac{\pi}{5}) = 1024$$
$$\left|X^{(1024)}[410]\right| = X_{tr}(\omega)\Big|_{\omega=\frac{2\pi\cdot 410}{1024}} \doteqdot X_{tr}(\tfrac{4}{5}\pi) = 256 \quad \ldots$$
$$\left|X^{(1024)}[614]\right| = 256$$
$$\left|X^{(1024)}[922]\right| = 1024$$



4. (29 pts) The N–point discrete Fourier Transform (DFT) is defined according to:

$$X^{(N)}[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}, \; k = 0, \ldots, N-1. \tag{1}$$

Suppose that you need to compute an exact 6-point DFT.

a) (12) Derive mathematical expressions that show how the 6-point DFT may be computed as a weighted sum of three 2-point DFTs of appropriately chosen subsequences of your 6-point data sequence. Be sure to show what is the weighting factor that is applied to each of the three 2-point DFTs. Also, be sure to show precisely which data-points are included in each of the three 2-point subsequences.

b) (8) Based on your answer to part a), draw a complete flow diagram for your 6-point Fast Fourier Transform (FFT) algorithm. Be sure to label all input points and output points in the diagram. Also, be sure to show all the weighting (twiddle) factors.

Define one complex operation (CO) as consisting of one complex addition and one complex multiplication.

c) (2) Determine approximately how many CO's are needed to compute the 6-point DFT by direct evaluation of Eq. (1).

d) (5) Based on a direct analysis of your flow diagram from part b), determine approximately how many CO's are needed to evaluate the 6-point DFT using your FFT algorithm.

e) (2) Suppose that you decide to *approximate* the output of the 6-point DFT by zero-padding your 6-point signal $x[n]$ to a length of 8 points; and then use a radix-2 FFT algorithm to compute the 8-point DFT of your zero-padded signal. How many CO's will this take?

(a) 6-pt DFT = 3 × 2-pt DFT

$$X^{(6)}[k] = \sum_{n=0}^{5} x[n] \cdot e^{-\frac{j2\pi kn}{6}} \qquad k = 0, \cdots 5$$

$$= \sum_{n=0}^{1} x[3n] e^{-\frac{j2\pi k3n}{6}} + \sum_{n=0}^{1} x[3n+1] \cdot e^{-\frac{j2\pi k(3n+1)}{6}}$$

$$+ \sum_{n=0}^{1} x[3n+2] e^{-\frac{j2\pi k(3n+2)}{6}}$$

$$= \sum_{n=0}^{1} x[3n] e^{-\frac{j2\pi kn}{2}} + \sum_{n=0}^{1} x[3n+1] e^{-\frac{j2\pi kn}{2}} \cdot e^{-\frac{j2\pi k}{6}}$$

$$+ \sum_{n=0}^{1} x[3n+2] e^{-\frac{j2\pi kn}{2}} \cdot e^{-\frac{j4\pi k}{6}}$$

4. (continued)

$$= X_0^{(2)}[k] + X_1^{(2)}[k]\cdot e^{-\frac{j2\pi k}{6}} + X_2^{(2)}[k]\cdot e^{-\frac{j4\pi k}{6}}$$

$$= X_0^{(2)}[k] + X_1^{(2)}[k]\cdot W_6^{k} + X_2^{(2)}[k]\cdot W_6^{2k}$$

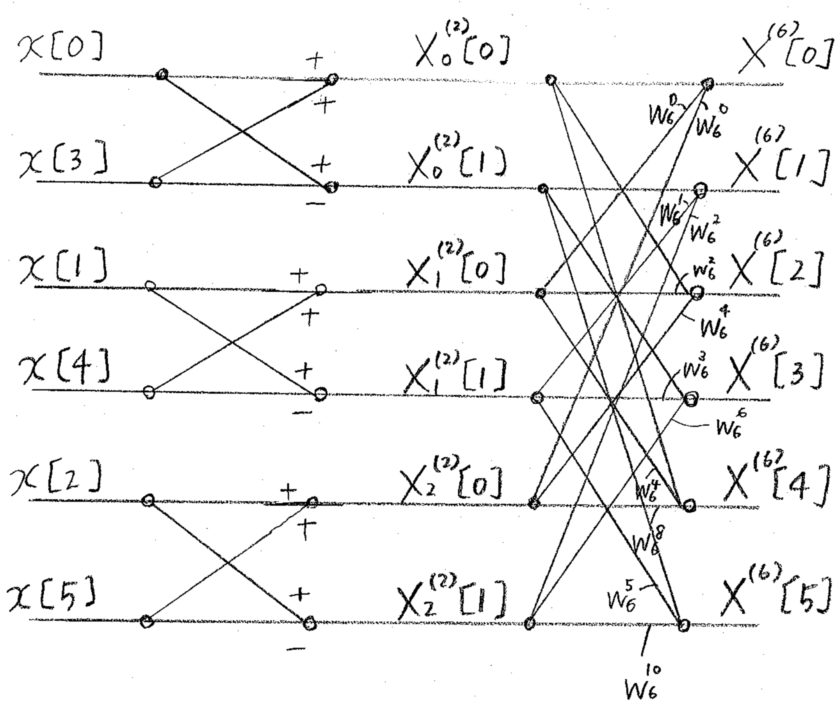
where

$$X_0^{(2)}[k] = DFT\{x[0], x[3]\}$$
$$X_1^{(2)}[k] = DFT\{x[1], x[4]\}$$
$$X_2^{(2)}[k] = DFT\{x[2], x[5]\}$$
$$W_6^{k} = e^{-\frac{j2\pi k}{6}}$$

(b) $$X_0^{(2)}[k] = \sum_{n=0}^{1} x[3n]\, e^{-\frac{j2\pi kn}{2}}$$

$$\therefore \begin{cases} X_0^{(2)}[0] = x[0] + x[3] \\ X_0^{(2)}[1] = x[0] - x[3] \end{cases},$$

$X_1^{(2)}[k]$ and $X_2^{(2)}[k]$ are similarly calculated.



4. (continued)

(c) $6^2 = 36$

(d) In the first stage, 6 additions/subtractions and no multiplies, so 0.5 co's $\times 6 = 3$ co's. (or just simply 6 co's)

In the second stage, $2 \times 6 = 12$ co's.

Ans. 15 co's (or 18)

(e) $N \log_2 N = 8 \cdot \log_2 8 = 24$

Ans. 24 co's

(This depends on how you count twiddle factors in radix 2 case.)