17 February 2010 **Name:** SOLUTION

ECE 438 **Exam No. 1** **Spring 2010**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n] = x[n] - x[n-1] - y[n-1].$$

Assume that the system is initially at rest, i.e. $y[n] = 0, n < 0$.

a. (6) For the following input signal $x[n]$, find the corresponding response $y[n]$ of the system.

n	…	−2	−1	0	1	2	3	4	5	…
$x[n]$	…	0	0	1	1	1	1	0	0	…

b. (6) Find a simple expression for the frequency response $H(\omega)$ of this system.

c. (6) Based on your answer to part (b), find simple expressions for the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response of this system.

d. (6) Based on your answer to part (c), carefully sketch the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response of this system.

e. (1) Is this system bounded-input-bounded-output (BIBO) stable? State why or why not.

(a)
$y[0] = 1 - 0 - 0 = 1$

$y[1] = 1 - 1 - 1 = -1$

$y[2] = 1 - 1 + 1 = 1$

$y[3] = 1 - 1 - 1 = -1$

$y[4] = 0 - 1 + 1 = 0$

$y[5] = 0 - 0 + 0 = 0$

(b) Let $x[n] = e^{j\omega n}$, $y[n] = H(\omega) e^{j\omega n}$

$\Rightarrow H(\omega) e^{j\omega n} = e^{j\omega n} - e^{j\omega(n-1)} - e^{j\omega(n-1)} H(\omega)$

$\Rightarrow H(\omega) e^{j\omega n} = e^{j\omega n} \left[1 - e^{-j\omega} - e^{-j\omega} H(\omega)\right]$

$\Rightarrow H(\omega)\left[1 + e^{-j\omega}\right] = 1 - e^{-j\omega}$

1. (continued)

$$\Rightarrow \quad H(\omega) = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}}\left[\frac{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}}\right]$$

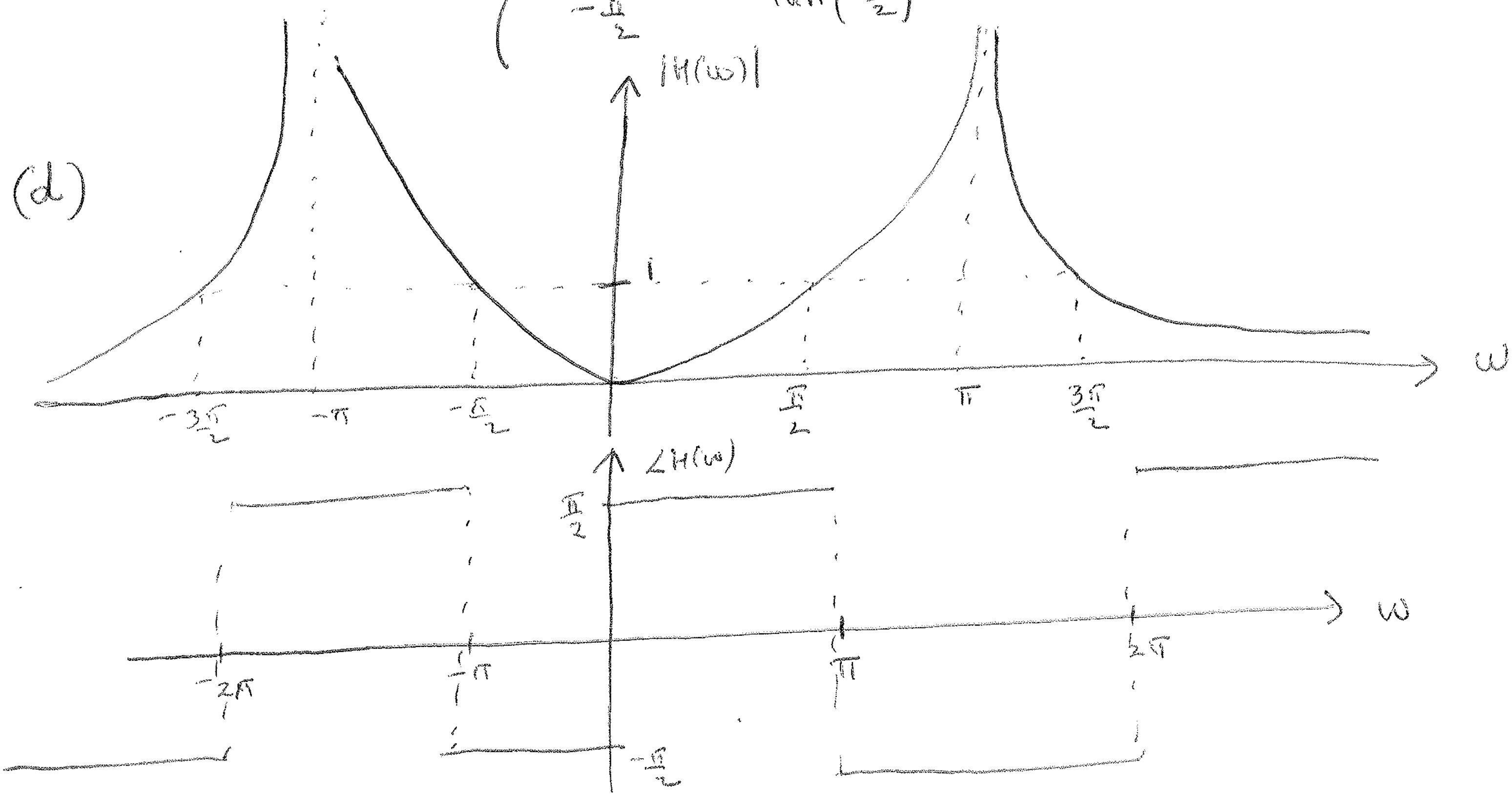
$$= \frac{2j}{2}\,\frac{\sin(\omega/2)}{\cos(\omega/2)} = j\tan\left(\frac{\omega}{2}\right)$$

(c) $|H(\omega)| = \left|\tan\left(\frac{\omega}{2}\right)\right|$

$$\angle H(\omega) = \angle j + \angle \tan\left(\frac{\omega}{2}\right)$$

$$= \begin{cases} \frac{\pi}{2} & \tan\left(\frac{\omega}{2}\right) \geq 0 \\ -\frac{\pi}{2} & \tan\left(\frac{\omega}{2}\right) < 0 \end{cases}$$

(d)



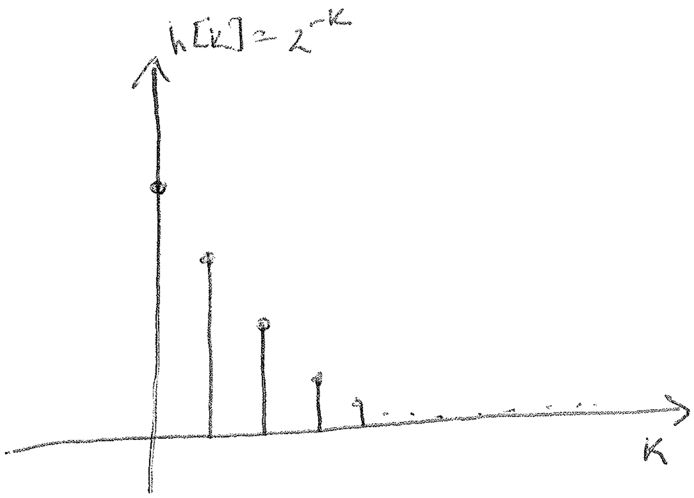
(e) No, since $|H(\omega)| \to \infty$ as $\omega \to \pi$

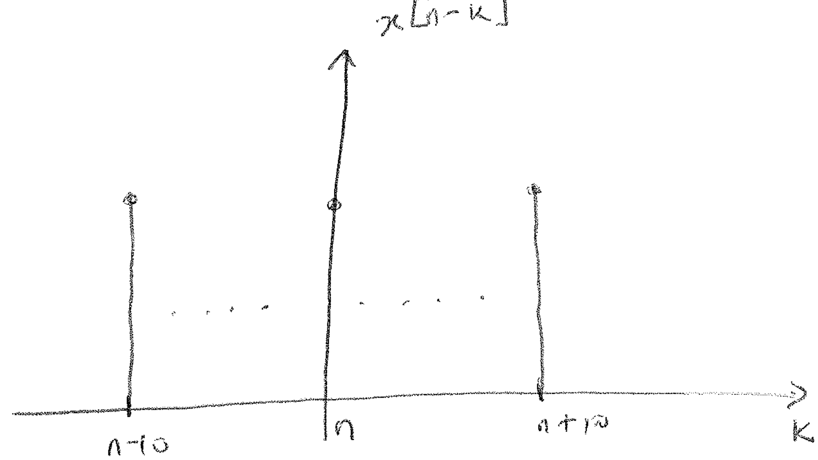
<u>OR</u>

$h[n] = \delta[n] + 2(-1)^n u[n-1]$

$\sum_{n=-\infty}^{\infty} |h[n]| = \infty$ so system is <u>NOT</u> BIBO stable!

2. (25 pts.) Consider a linear, shift-invariant system with impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$. Find an expression for the response $y[n]$ to the input $x[n] = \begin{cases} 1, & |n| \le 10 \\ 0, & \text{else} \end{cases}$ by evaluating the convolution sum $y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$. Simplify your answer as much as possible.





Case 1: $n+10 < 0 \Rightarrow n < -10$

$y[n] = 0$, since no overlap between h and x

Case 2: $n+10 \geq 0$ and $n-10 \leq 0 \Rightarrow -10 \leq n \leq 10$

$$y[n] = \sum_{k=0}^{n+10} \left(\frac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^{n+11}}{1-\frac{1}{2}} = 2\left[1-\left(\frac{1}{2}\right)^{n+11}\right]$$

Case 3: $n-10 > 0 \Rightarrow n > 10$

$$y[n] = \sum_{k=n-10}^{n+10} \left(\frac{1}{2}\right)^k, \quad \text{let } m = k-(n-10)$$

$$\Rightarrow y[n] = \sum_{m=0}^{20} \left(\frac{1}{2}\right)^{m+(n-10)} = \left(\frac{1}{2}\right)^{n-10} \sum_{m=0}^{20} \left(\frac{1}{2}\right)^m$$

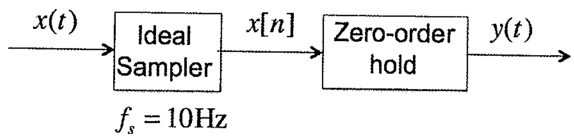
$$= \left(\frac{1}{2}\right)^{n-10}\left[\frac{1-\left(\frac{1}{2}\right)^{21}}{1-\frac{1}{2}}\right]$$

$$= \left(\frac{1}{2}\right)^n\left[2^{11}-2^{-10}\right]$$

2. (continued)

$$\therefore \quad y[n] = \begin{cases} 0 & n < -10 \\ 2\left[1 - \left(\frac{1}{2}\right)^{n+11}\right] & -10 \le n \le 10 \\ \left(\frac{1}{2}\right)^{n}\left[2^{11} - 2^{-10}\right] & n > 10 \end{cases}$$

3. (25) Consider the system shown in the block diagram below:



The operation of the zero-order hold unit can be described as:

$$y(t) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{rect}\left(\frac{t - 0.1n - 0.05}{0.1}\right).$$

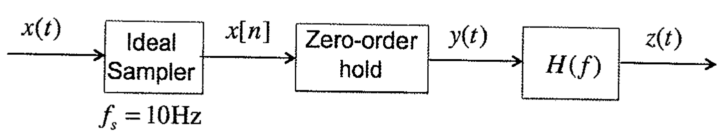
Suppose that the input to this system is $x(t) = \cos(2\pi(10/8)t)$.

a. (8) Carefully and accurately sketch the output $y(t)$ of the zero-order hold.

b. (9) Find an expression for the continuous-time Fourier transform (CTFT) $Y(f)$ of the output $y(t)$ for the specific input $x(t) = \cos(2\pi(10/8)t)$ to the system. Your answer should be simplified as much as possible, and should not contain any operators.

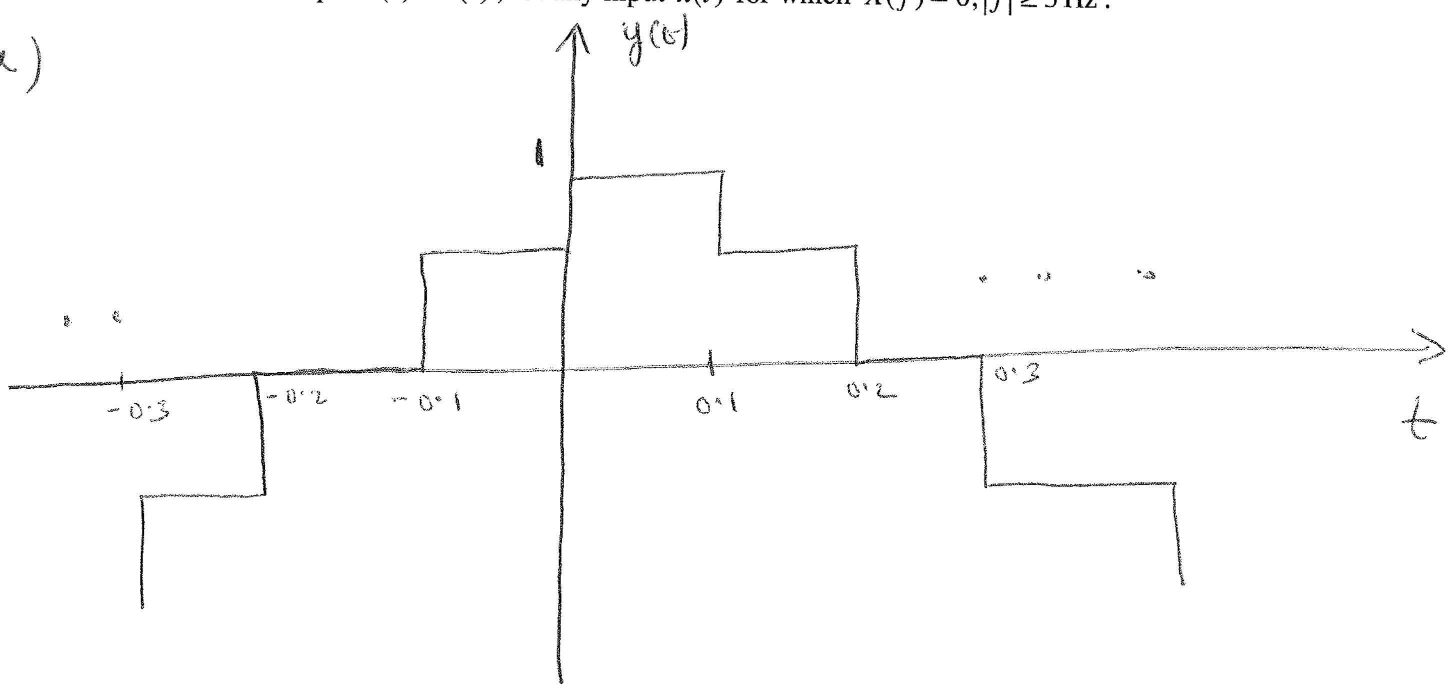
c. (4) Sketch $Y(f)$.

Now consider placing an analog filter with frequency response $H(f)$ at the output of the system shown above to yield the following system:



d. (4) Find the desired frequency response $H(f)$ of the filter so that the system output $z(t) \equiv x(t)$, for any input $x(t)$ for which $X(f) = 0, |f| \geq 5\,\text{Hz}$.

(a)



3. (continued)

(b)

$$x(t) = \cos\left(2\pi\left(\tfrac{10}{8}\right)t\right)$$

$$x_s(t) = \underset{0.1}{\text{comb}}\; x(t)$$

$$x_r(t) = \text{rect}\left(\frac{t-0.05}{0.1}\right)$$

So, we have $y(t) = x_s * x_r(t)$

$$X(f) = \frac{1}{2}\left[\delta\left(f-\tfrac{10}{8}\right) + \delta\left(f+\tfrac{10}{8}\right)\right]$$

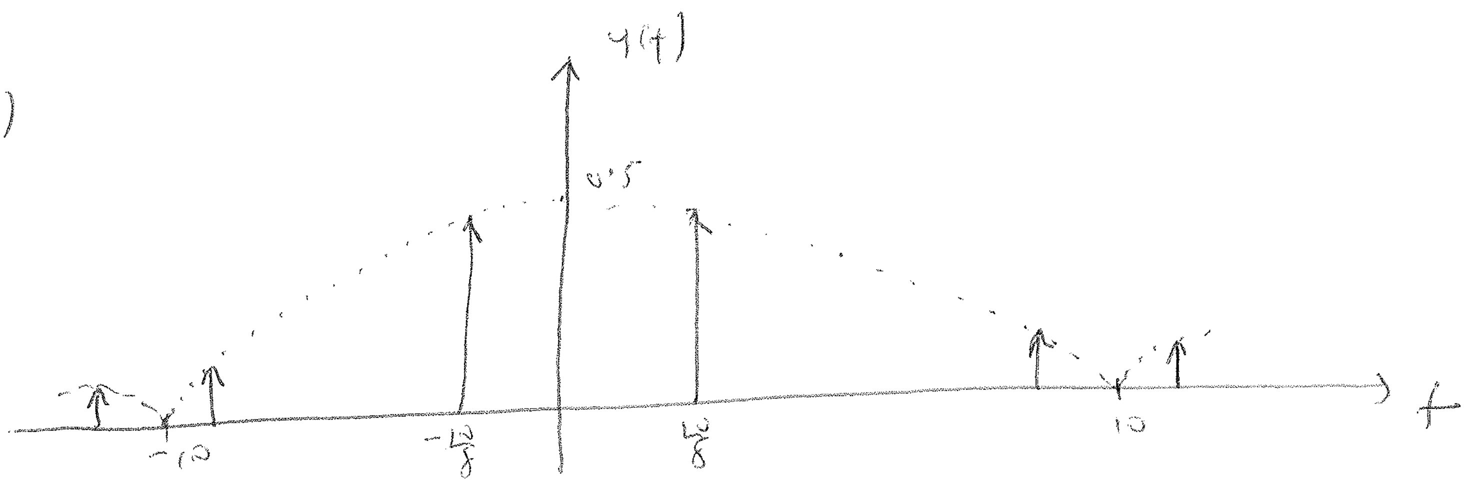
$$X_s(f) = \frac{1}{0.1}\underset{\frac{1}{0.1}}{\text{rep}}\; X(f) = 10\,\text{rep}_{10}\left\{\frac{1}{2}\left[\delta\left(f-\tfrac{10}{8}\right) + \delta\left(f+\tfrac{10}{8}\right)\right]\right\}$$

$$X_r(f) = 0.1\,\text{sinc}(0.1f)\,e^{-j2\pi f(0.05)}$$

$$\Rightarrow \quad Y(f) = X_s(f)\,X_r(f)$$

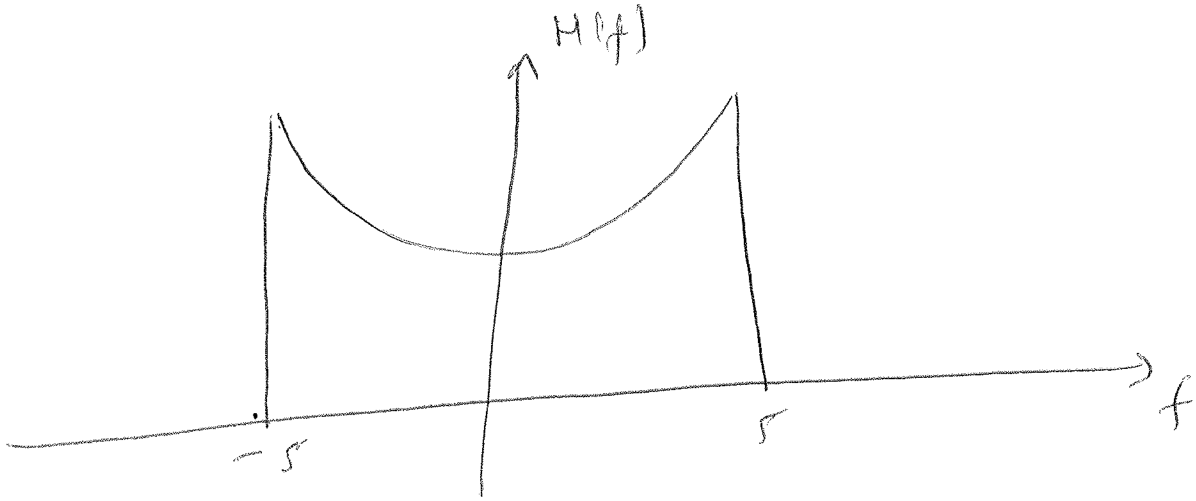
$$= \text{rep}\left\{\frac{1}{2}\left[\delta\left(f-\tfrac{10}{8}\right) + \delta\left(f+\tfrac{10}{8}\right)\right]\right\}\text{sinc}(0.1f)\,e^{-j2\pi f(0.05)}$$

(c)

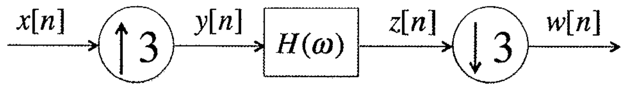


3. (continued)

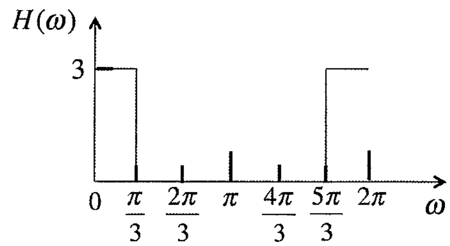
(d) $H(f) = \begin{cases} \dfrac{1}{\text{sinc}(0.1f)\, e^{-j2\pi f(0.05)}} & -5 \leq f \leq 5 \\ 0 & \text{otherwise} \end{cases}$



4. (25 pts) Consider the system shown below:



Here the filter $H(\omega)$ has the frequency response:



Let uppercase symbols denote the DTFTs of their corresponding lowercase quantities.

a) (3) Find a simple expression for $W(\omega)$ in terms of $Z(\omega)$.

b) (3) Find a simple expression for $Z(\omega)$ in terms of $Y(\omega)$.

c) (3) Find a simple expression for $Y(\omega)$ in terms of $X(\omega)$.

d) (8) Combine your answers to parts (a) – (c) above to find a simple expression for $W(\omega)$ in terms of $X(\omega)$.

e) (8) Suppose $X(\omega)$ is given by $X(\omega) = 1 - \left|\frac{\omega}{\pi}\right|, 0 \le |\omega| \le \pi$. Sketch $X(\omega)$, $Y(\omega)$, $Z(\omega)$, and $W(\omega)$.

(a) $W(\omega) = \frac{1}{3}\sum_{k=0}^{2} Z\left(\frac{\omega - 2\pi k}{3}\right)$

(b) $Z(\omega) = Y(\omega)\, H(\omega)$

(c) $Y(\omega) = X(3\omega)$

(d) $W(\omega) = \frac{1}{3}\sum_{k=0}^{2} Y\left(\frac{\omega - 2\pi k}{3}\right) H\left(\frac{\omega - 2\pi k}{3}\right)$

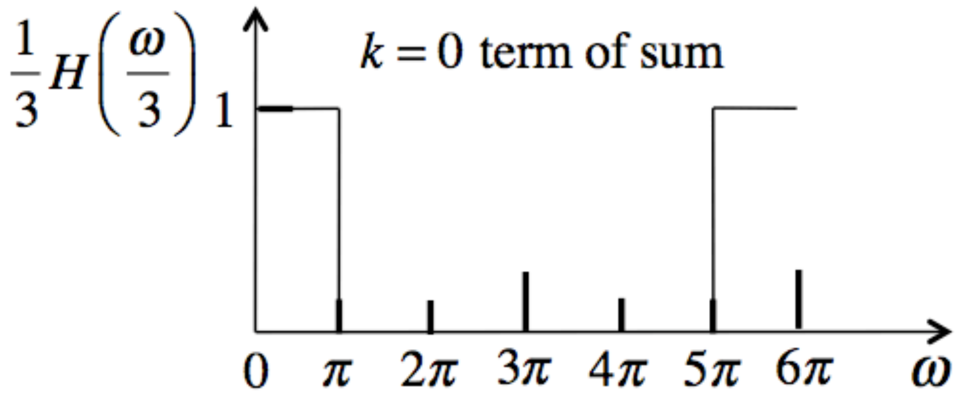
$= \frac{1}{3}\sum_{k=0}^{2} X(\omega - 2\pi k)\, H\left(\frac{\omega - 2\pi k}{3}\right)$

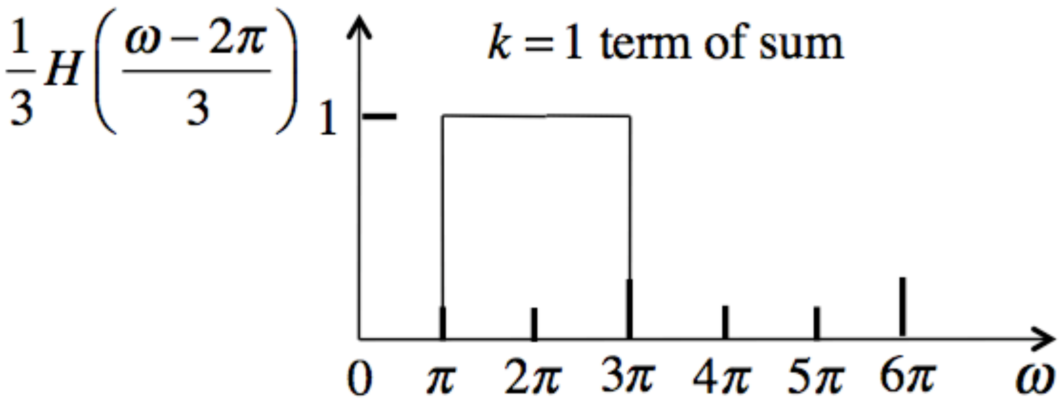
After further simplification

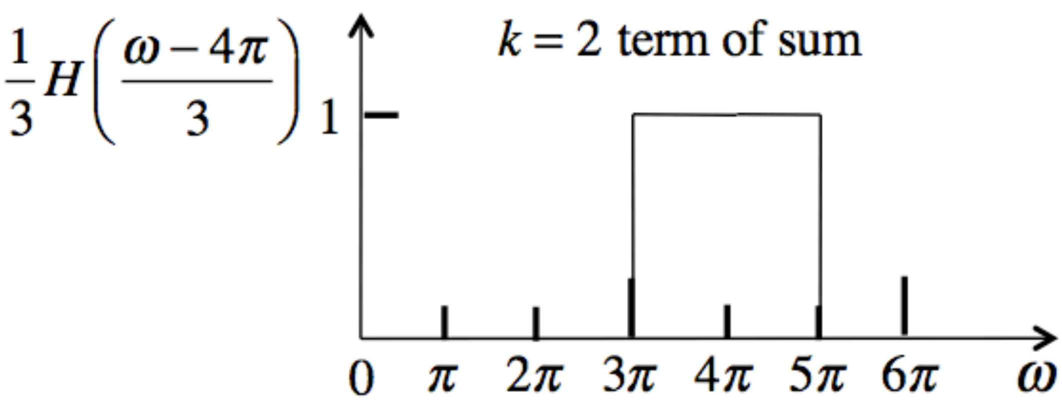
$W(\omega) = X(\omega)$

Specifically, since $X(\omega - 2\pi k) = X(\omega)$ for any integer k, we have $W(\omega) = X(\omega)\frac{1}{3}\sum_{k=0}^{2} H\left(\frac{\omega - 2\pi k}{3}\right)$.

But $\frac{1}{3}\sum_{k=0}^{2} H\left(\frac{\omega - 2\pi k}{3}\right) = 1$. See figure on next page.


$\frac{1}{3}H\left(\frac{\omega}{3}\right)$
1
$k = 0$ term of sum
0
π
2π
3π
4π
5π
6π
ω


$\frac{1}{3}H\left(\frac{\omega - 2\pi}{3}\right)$
1
$k = 1$ term of sum
0
π
2π
3π
4π
5π
6π
ω


$\frac{1}{3}H\left(\frac{\omega - 4\pi}{3}\right)$
1
$k = 2$ term of sum
0
π
2π
3π
4π
5π
6π
ω

4. (continued)

(e)