28 April 2008 **Name:** Key

ECE 438 **Final Exam** **Spring 2008**

- You have 120 minutes to work the following *four* problems. Each problem is worth 30 points.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (30 pts.) Consider the two different digital systems described by the two input-output relationships shown below. For each system, please answer the following questions:

   i. (1) Is this system linear?
   ii. (1) Is this system time-invariant?
   iii. (1) Is this system causal?
   iv. (1) Is this system memoryless?
   v. (1) Is this system BIBO stable?
   vi. (5) Find the response $y[n]$ of the system to the following input

$$x[n] = \begin{cases} 0, & n<0 \\ 1, & n=0,1 \\ 4, & n=2,3,4 \\ 9, & n=5,6,7,8,9 \\ 0, & n>9 \end{cases}$$

   vii. (5) Find simple expressions for the magnitude and phase of the frequency response of the system. Here we define the frequency response $H(\omega)$ as the ratio of the output $y[n]$ to the input $x[n]$, when the input is $x[n]=e^{j\omega n}$.

   Note: Your answers to parts i) through v) should just be "yes" or "no". No justification is needed, and no partial credit will be given.

   a. (15) System 1 (arithmetic mean): $y[n]=\frac{1}{2}(x[n]+x[n-1])$

   b. (15) System 2 (geometric mean): $y[n]=(x[n]\bullet x[n-1])^{\frac{1}{2}}$

(a) $y[n]=\frac{1}{2}(x[n]+x[n-1])$

(i) yes, linear

(ii) yes, time-invariant

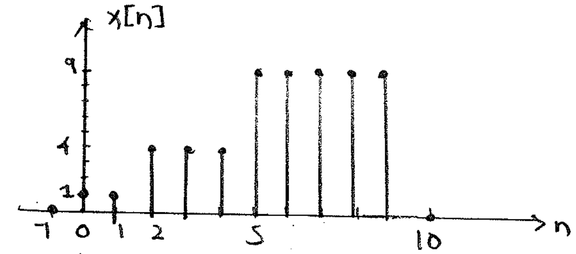
(iii) yes, causal

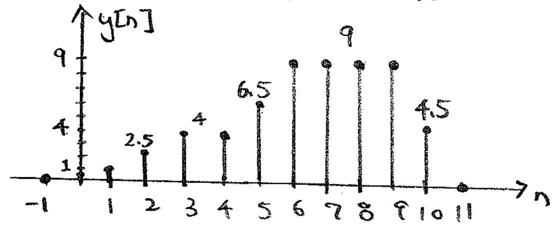
(iv) no, not memoryless

(v) yes, BIBO stable

1. (continued)

(a) (vi)





$y[n] = \{\ldots, 0, \underset{\substack{\uparrow \\ n=0}}{0.5}, 1, 2.5, 4, 4, 6.5, 9, 9, 9, 9, \underset{\substack{\uparrow \\ n=10}}{4.5}, 0, \ldots\}$

(vii) let $x[n] = e^{j\omega n}$, $y[n] = H(e^{j\omega})e^{j\omega n}$

$$H(e^{j\omega})e^{j\omega n} = \frac{1}{2}\left(e^{j\omega n} + e^{j\omega(n-1)}\right)$$

$$H(e^{j\omega})\cancel{e^{j\omega n}} = \frac{1}{2}\cancel{e^{j\omega n}}\left(1 + e^{-j\omega}\right)$$

$$H(e^{j\omega}) = \frac{1}{2}(1+e^{-j\omega}) = \boxed{e^{-j\frac{\omega}{2}}\cos\left(\frac{\omega}{2}\right)}$$

$$\angle H(e^{j\omega}) = \begin{cases} -\frac{\omega}{2}, & \cos\left(\frac{\omega}{2}\right) > 0 \\ -\frac{\omega}{2} \pm \pi, & \cos\left(\frac{\omega}{2}\right) < 0 \end{cases}$$

$$|H(e^{j\omega})| = \left|\cos\left(\frac{\omega}{2}\right)\right|$$

(b) $y[n] = (x[n] \cdot x[n-1])^{\frac{1}{2}}$

(i) no, not linear

(ii) yes, time-invariant

(iii) yes, causal

(iv) no, not memoryless

(v) yes, BIBO stable

1. (continued)

(b) (vi) $y[n] = \{\cdots, \underset{\uparrow n=0}{0}, 1, 2, 4, 4, 6, 9, 9, 9, 9, \underset{\uparrow n=10}{0}, \cdots\}$

(vii) let $x[n] = e^{j\omega n}$, $y[n] = H(e^{j\omega})e^{j\omega n}$

$H(e^{j\omega})e^{j\omega n} = (e^{j\omega n} e^{j\omega(n-1)})\frac{1}{2}$

$H(e^{j\omega})\cancel{e^{j\omega n}} = \cancel{e^{j\omega n}} e^{-j\frac{\omega}{2}}$

$H(e^{j\omega}) = e^{-j\frac{\omega}{2}}$

$\angle H(e^{j\omega}) = -\frac{\omega}{2}$

$|H(e^{j\omega})| = 1$

2. (30 pts.) FFT and DFT

a. (20) Suppose that you wish to build a hardware system that will compute the exact DFT of a length 640-point signal. You have at your disposal a chip that will compute a length 128-point FFT.

i. (10) Derive a set of equations that show exactly how to efficiently compute the 640-point DFT using one or more of the 128-point FFT chips.

ii. (5) Draw a block diagram for your system being careful to show enough detail to clearly indicate the overall structure.

iii. (2) Determine approximately how many complex operations would be required to evaluate the 640-point DFT directly.

iv. (3) Determine approximately how many complex operations would be required to evaluate the 640-point DFT using your system with the 128-point FFT chips.

Here a *complex operation* consists of one complex multiplication and one complex addition.

b. (10) Suppose that you have an N-point signal $x[n],\ n=0,\ldots,N-1$, and that you create a $2N$-point signal $y[n],\ n=0,\ldots,2N-1$ by simply repeating $x[n]$; so

$$y[n]=\begin{cases} x[n], & n=0,\ldots,N-1 \\ x[n-N] & n=N,\ldots,2N-1 \end{cases}$$

Find a simple expression for the $2N$-point DFT $Y^{2N}[k],\ k=0,\ldots,2N-1$ of $y[n]$ in terms of the N-point DFT $X^{N}[k],\ k=0,\ldots,N-1$ of $x[n]$.

(a) (i) $$X^{(N)}(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{l=0}^{4} \sum_{n=0}^{127} x[5n+l] e^{-j\frac{2\pi k(5n+l)}{640}}$$

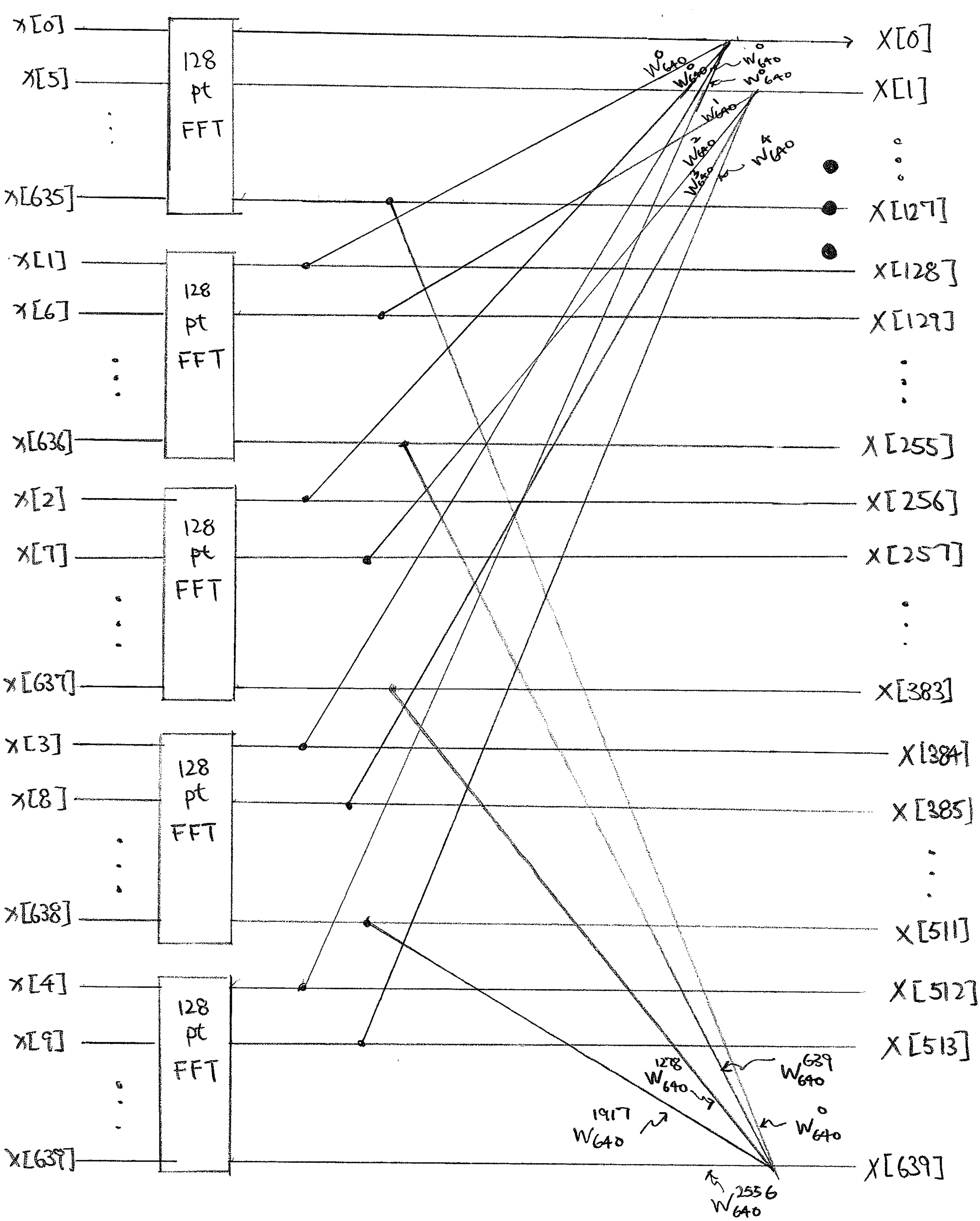
$$= \sum_{l=0}^{4} e^{-j\frac{2\pi kl}{640}} \sum_{n=0}^{127} x[5n+l] e^{-j 2\pi kn/128}$$

$$= X_0^{(128)}[k] + e^{-j\frac{2\pi k}{640}} X_1^{(128)}[k] + e^{-j\frac{4\pi k}{640}} X_2^{(128)}[k] \ ?$$

$$+ e^{-j\frac{6\pi k}{640}} X_3^{(128)}[k] + e^{-j\frac{8\pi k}{640}} X_4^{(128)}[k]$$

2. (continued)

(a) (ii)



2. (continued)

(a) (iii) direct DFT

$$N^2 = 640^2 = 409600 \text{ complex operations}$$

(iv) each 128-point FFT block:

$$M \log M = 128 \log 128 = 128 \cdot 7 = 896 \text{ complex operations}$$

combination stage:

$$4 \times 640 = 2560 \text{ complex operations}$$

4 twiddle factor multiplications and 4 adds for each $X[k]$

Total: $5 \cdot 896 + 2560 = 4480 + 2560$

$= 7040$ complex operations

(b) $$Y^{2N}[k] = \sum_{n=0}^{2N-1} y[n] e^{-j\frac{2\pi kn}{2N}}$$

for k even: $k = 2r$, $r = 0, 1, \ldots, N-1$

$$Y[2r] = \sum_{n=0}^{2N-1} y[n] W_{2N}^{2rn} = \sum_{n=0}^{N-1} y[n] W_{2N}^{2rn} + \sum_{n=0}^{N-1} y[n+N] W_{2N}^{2r(n+N)}$$

$$= \sum_{n=0}^{N-1} \Big\{ [\underbrace{y[n]}_{=x[n]} + \underbrace{y[n+N]}_{=x[n]}] W_{\frac{2N}{2}}^{rn} \Big\} \quad \text{by definition of } y[n]$$

$$= 2 \sum_{n=0}^{N-1} x[n] W_N^{rn} = 2X[\tfrac{k}{2}] \quad \text{for } k = 0, 2, 4, \ldots, 2N-2$$

for k odd: $k = 2r+1$, $r = 0, 1, \ldots, N-1$

$$Y[2r+1] = \sum_{n=0}^{2N-1} y[n] W_N^{(2r+1)n} = \sum_{n=0}^{N-1} \Big\{ (y[n] + \cancel{W_N^{\frac{N}{2}}}^{-1} y[n+N]) W_{2N}^{n} \Big\} W_N^{rn}$$

$$= \sum_{n=0}^{N-1} \Big\{ (\cancel{x[n] - x[n]}^{\,0}) W_{2N}^{n} \Big\} W_N^{rn} = 0$$

$$\therefore Y[k] = \begin{cases} 2X[\frac{k}{2}] & \text{for } k = 0, 2, 4, \ldots, N-2 \\ 0 & \text{for } k = 1, 3, 5, \ldots, N-1 \end{cases}$$

3. (30) The short-time Fourier transform of the signal $s(t)$ can also be defined for continuous time as follows:

$$S(f,t) = \int s(\tau) w(t-\tau) e^{-j2\pi f \tau} d\tau .$$

Here, the first argument f in the short-time continuous-time Fourier transform (STCTFT) $S(f,t)$ denotes the short-time frequency of the signal $s(\tau)$ in units of Hertz and the second argument t denotes the location in units of seconds of the center of the window function $w(-(\tau - t))$ that is used to time-limit the speech signal $s(\tau)$.

Consider the specific signal

$$s(t) = \text{rep}_{0.01}\left[\cos(2\pi(1000t))\text{rect}(t/0.005)\right],$$

that may be regarded as a simple model for a voiced speech waveform that contains just one formant frequency. (Here t is in units of seconds.)

a. (6) *Carefully* sketch the waveform $s(t)$, being sure that your sketch is accurate, and that you have labeled the dimension of each important quantity.

b. (2) What are the pitch period and formant frequency for this waveform?

c. (8) Assume that the window function is $w(\tau) = \text{rect}(\tau / 0.005)$. Compute the exact STCTFT $S(f,t)$ for the times $t = 0$ and $t = 0.005$.

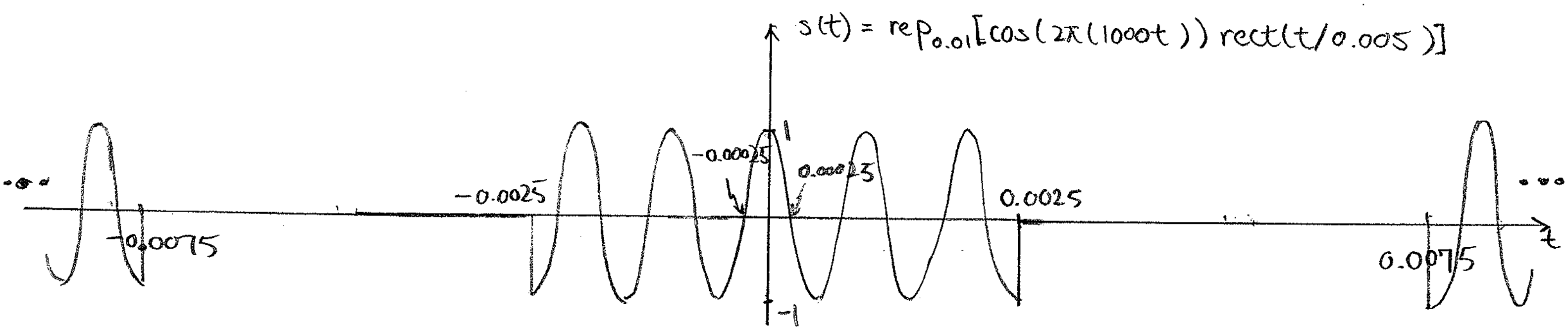
d. (3) Based on your answer to part c) above, approximately sketch the spectrogram $S(f,t)$ for all times t, being sure to label all important quantities. Is this spectrogram narrowband or wideband?

e. (8) Assume that the window function is $w(\tau) = \text{rect}(\tau / 0.1)$. Compute the exact STCTFT $S(f,t)$ for the time $t = 0$.

f. (3) Based on your answer to part e) above, approximately sketch the spectrogram $S(f,t)$ for all times t, being sure to label all important quantities. Is this spectrogram narrowband or wideband?

Note: In working this problem, you should use, as much as possible, the standard transform relations and transform pairs for the CTFT that can be found on the formula sheet.

(a) $\text{rect}(t/0.005) = \begin{cases} 1 & \left|\frac{t}{0.005}\right| < \frac{1}{2} \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & |t| < 0.0025 \\ 0 & \text{else} \end{cases}$



3. (continued)

(b) pitch period : $T_p = 0.01$ second $= 10$ ms

pitch freq: $f_p = \frac{1}{T_p} = 100$ Hz , formant freq: $F = 1000$ Hz

(c) $S(f,t) = \int \text{rep}_{0.01}[\cos(2\pi(1000\tau))\text{rect}(\tau/0.005)]\,\text{rect}((t-\tau)/0.005)e^{-j2\pi f\tau}d\tau$

for t=0:

$$S(f,0) = \int \underbrace{\cos(2\pi(1000\tau))\,\overbrace{\text{rect}(-\tau/0.005)}^{=\text{rect}(\tau/0.005) \text{ by symmetry}}}_{(*)} e^{-j2\pi f\tau}d\tau$$

⇒ recognize that $S(f,0)$ is the Fourier transform of (*)

note: $\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}\{\delta(f-f_0)+\delta(f+f_0)\}$

$\text{rect}(\frac{t-t_0}{a}) \leftrightarrow |a|\text{sinc}(af)e^{-j2\pi f t_0}$

$x(t)y(t) \leftrightarrow X(f)*Y(f)$

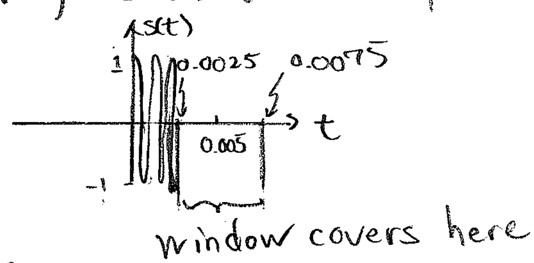
using these relations

$$\therefore\ S(f,0) = \frac{1}{2}\{\delta(f-1000)+\delta(f+1000)\} * 0.005\,\text{sinc}(0.005f)$$

$$= 0.0025\{\text{sinc}(0.005(f-1000)) + \text{sinc}(0.005(f+1000))\}$$

for t=0.005

$$S(f,0.005) = \int \underbrace{\overbrace{\text{rep}_{0.01}[\cos(2\pi(1000\tau))\text{rect}(\tau/0.005)]}^{s(\tau)}\text{rect}(\tfrac{0.005-\tau}{0.005})}_{\text{the window falls on the zero portion of } s(t)} e^{-j2\pi f\tau}d\tau$$

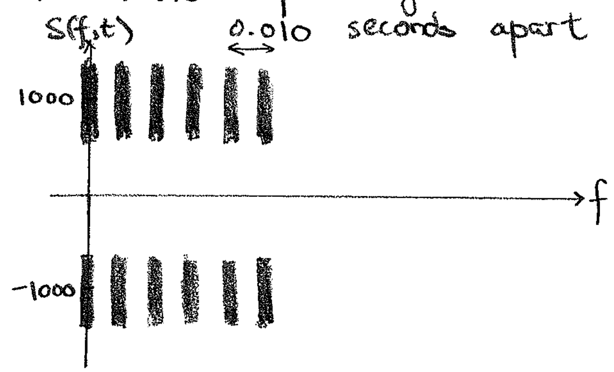


$$= \int (0)e^{-j2\pi f\tau}d\tau$$

$$= 0$$

3. (continued)

3. (d) wideband spectrogram



(e) $t = 0, \quad w(\tau) = \text{rect}(t/0.1)$

$$S(f,0) = \int \text{rep}_{0.01}[\cos(2\pi(1000\tau))\text{rect}(\tau/0.005)]\,\text{rect}\left(\frac{-\tau}{0.1}\right) e^{-j2\pi f\tau}\, d\tau$$

similar to part (c) but must use relation:

$$\text{rep}_T[x(t)] \longleftrightarrow \frac{1}{T}\text{comb}_{\frac{1}{T}}[X(f)]$$

recall: $v(t) = \cos(2\pi(1000t))\,\text{rect}(t/0.005)$, note: $s(t) = \text{rep}_{0.01}[v(t)]$

$$\updownarrow$$

$$V(f) = 0.0025\{\text{sinc}(0.005(f-1000)) + \text{sinc}(0.005(f+1000))\}$$

$$S(f,0) = \frac{1}{0.01}\text{comb}_{\frac{1}{0.01}}[V(f)] * 0.1\,\text{sinc}(0.1f)$$

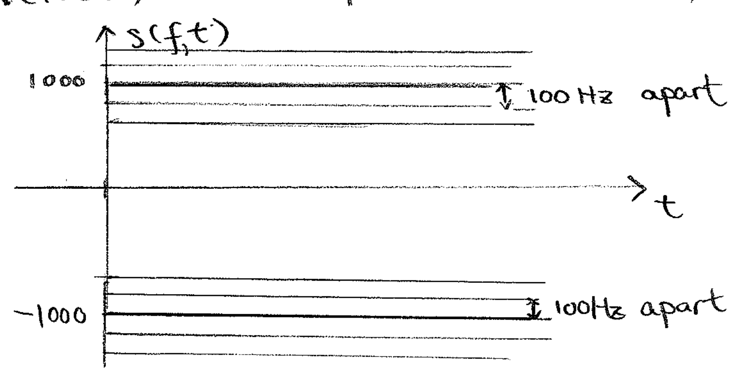
$$= \left\{100\sum_{k=-\infty}^{\infty} V(100k)\,\delta(f-100k)\right\} * 0.1\,\text{sinc}(0.1f)$$

$$= 10\sum_{k=-\infty}^{\infty} V(100k)\{\delta(f-100k) * \text{sinc}(0.1f)\}$$

$$= 10\sum_{k=-\infty}^{\infty} V(100k)\,\text{sinc}(0.1(f-100k))$$

where $V(100k) = 0.0025\{\text{sinc}(0.005(100k-1000)) + \text{sinc}(0.005(100k-1000))\}$

(f) narrowband spectrogram



4. (30 pts) Consider a spatial filter with point spread function $h[m,n]$ given below

$$h[m,n] \qquad \begin{array}{c|ccc} m \backslash n & -1 & 0 & 1 \\ \hline -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{array}$$

a. (13) Find the output $g[m,n]$ when this filter is applied to the following input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 11×11 set of pixels in the input image.

0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

b. (12) Find a simple expression for the frequency response $H(\mu, \nu)$ of this filter, and sketch the magnitude $|H(\mu,\nu)|$ along the μ axis, the ν axis, the $\mu = \nu$ axis, and the $\mu = -\nu$ axis. (*Note:* This filter is not separable. You just have to compute the 2-D Discrete-space Fourier transform directly.)

c. (5) Using your results from parts a) and b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

4. (continued)

(a)

-1	-1	0	0	0	1	1	0	0	0	0
-1	0	1	1	1	1	1	1	0	0	0
0	1	1	1	1	0	0	1	1	0	0
0	1	1	1	1	1	0	0	1	1	0
0	1	1	1	1	1	1	0	0	1	1
0	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	2	2	-1	-1
0	1	1	1	1	1	2	2	-1	-1	0
0	1	1	1	1	2	2	-1	-1	0	0
1	2	1	1	1	1	-1	-1	0	0	0
1	1	0	0	0	-1	-1	0	0	0	0

(b) $h[m,n] = -\delta(m+1,n+1) + \delta(m+1,n-1) + \delta(m,n) + \delta(m-1,n+1) - \delta(m-1,n-1)$

$$H(u,v) = -e^{j(u+v)} + e^{j(u-v)} + 1 + e^{j(-u+v)} - e^{-j(u+v)}$$

along u axis : $v=0$

$$|H(u,0)| = \left| \cancel{-e^{ju}} \cancel{+e^{ju}} + 1 + \cancel{e^{-ju}} \cancel{-e^{-ju}} \right|$$

$$= 1$$

along v axis : $u=0$

$$|H(0,v)| = \left| \cancel{-e^{jv}} \cancel{+e^{jv}} + 1 + \cancel{e^{-jv}} \cancel{-e^{-jv}} \right|$$

$$= 1$$

along $u=v$ axis :

$$|H(u,u)| = \left| -e^{j2u} + \cancelto{1}{e^{j(0)}} + 1 + \cancelto{1}{e^{j(0)}} - e^{-j2u} \right|$$

$$= |3 - 2\cos(2u)| = 3 - 2\cos(2u)$$

along $u=-v$ axis :

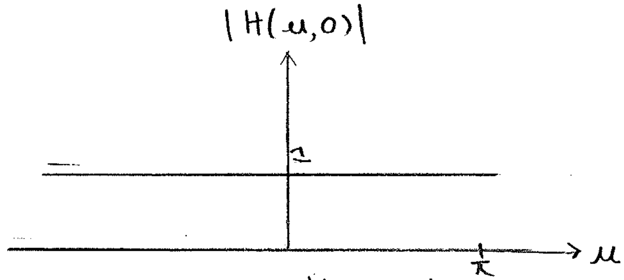
$$|H(u,-u)| = \left| -\cancelto{1}{e^{j0}} + e^{j2u} + 1 + e^{-j2u} - \cancelto{1}{e^{-j(0)}} \right|$$

$$= |-1 + 2\cos(2u)|$$

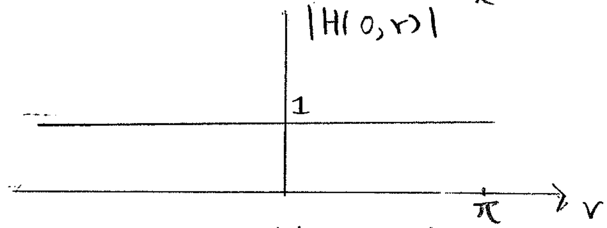
graphs on next page

4. (continued)

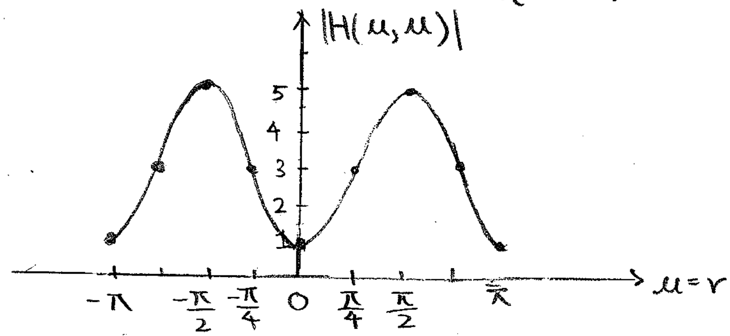
along μ axis



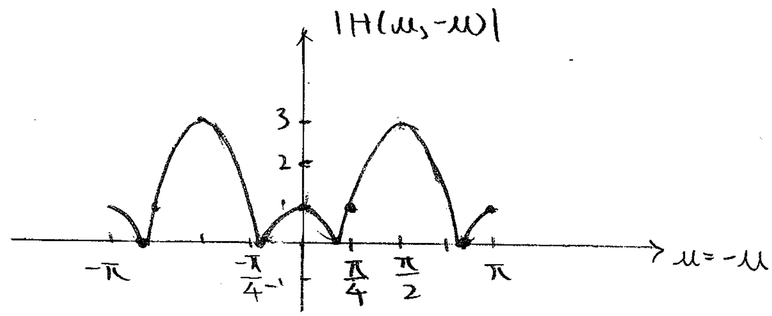
along ν axis



along $\mu = \nu$ axis



along $\mu = -\nu$ axis



(b) In image domain, the filter does not affect frequency components along either vertical or horizontal directions. Along the $m=n$ diagonal, this filter emphasizes edges by increasing mid-frequency components. Along the $m=-n$ diagonal, this filter also emphasizes edges by increasing mid-frequency components, but attenuates frequencies around $\pi/6$ and $5\pi/6$. Note that the block of 1's is maintained after filtering with no effects introduced in its center. This is "DC-preserving" since the sum of filter coefficients is 1 and $H(0,0) = 1$.

1.	______
2.	______
3.	______
4.	______
Total	______