19 March 2008 **Name:** Key

ECE 438 **Exam No. 2** **Spring 2008**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider a causal linear, time-invariant system with transfer function

$$H(z) = \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}.$$

a. (3) Find the region of convergence for $H(z)$. Is this system BIBO stable?
b. (13) Find the impulse response $h[n]$.
c. (9) Use the graphical approach to find the magnitude and phase of the frequency response at the frequency $\omega = \pi/2$ radians/sample.

(a) given system is causal, poles located at $\frac{1}{2}$ and $-\frac{1}{2}$

ROC: $|z| > \frac{1}{2}$

Since the ROC includes the unit circle, system is BIBO stable.

(b) $H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$

$A = H(z)(1+\frac{1}{2}z^{-1})\Big|_{z=-\frac{1}{2}} = \frac{1}{1-\frac{1}{2}z^{-1}}\Big|_{z=-\frac{1}{2}} = \frac{1}{1-\frac{1}{2}(-2)} = \frac{1}{2}$

$B = H(z)(1-\frac{1}{2}z^{-1})\Big|_{z=\frac{1}{2}} = \frac{1}{1+\frac{1}{2}z^{-1}}\Big|_{z=\frac{1}{2}} = \frac{1}{2}$

$\Rightarrow H(z) = \frac{1}{2}\frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1}{2}\frac{1}{1-\frac{1}{2}z^{-1}}$

Since system is causal, $h[n]$ is right sided

$h[n] = \frac{1}{2}(-\frac{1}{2})^n u[n] + \frac{1}{2}(\frac{1}{2})^n u[n]$

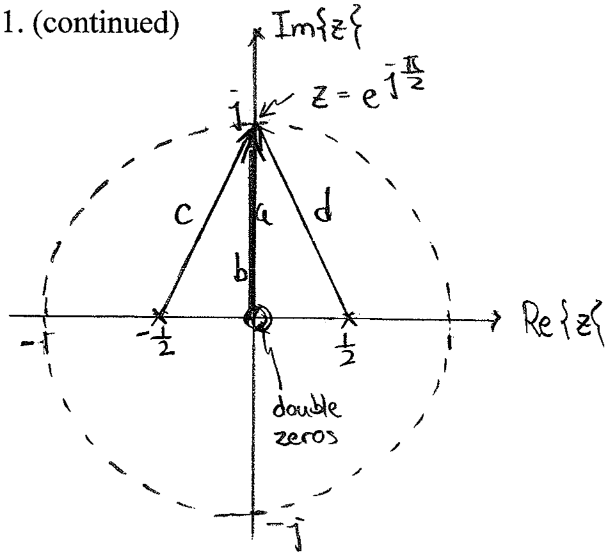
$= \frac{1}{2}u[n]\left\{(-\frac{1}{2})^n + (\frac{1}{2})^n\right\}$

$= \begin{cases} \frac{1}{2}u[n]\left\{2(\frac{1}{2})^n\right\} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

$= \begin{cases} (\frac{1}{2})^n u[n] & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

1. (continued)

(c)



$$H(z) = \frac{z^2}{(z+\frac{1}{2})(z-\frac{1}{2})}$$

double zeros at $z = 0$

poles at $z = \pm\frac{1}{2}$

$$|H(e^{j\frac{\pi}{2}})| = \frac{|a||b|}{|c||d|}, \quad |a| = |b| = 1, \quad |c| = |d| = \sqrt{(\tfrac{1}{2}-0)^2+(0-1)^2} = \sqrt{\tfrac{5}{4}}$$

$$= \frac{1}{\sqrt{\frac{5}{4}}\sqrt{\frac{5}{4}}}$$

$$= \frac{4}{5} = 0.8$$

$$\angle H(e^{j\frac{\pi}{2}}) = \angle a + \angle b - \angle c - \angle d$$

note : $\angle a = \angle b = \frac{\pi}{2}$

$\angle d = \pi - \angle c$

$$\therefore H(e^{j\frac{\pi}{2}}) = \frac{\pi}{2} + \frac{\pi}{2} - \cancel{\angle c} - \pi + \cancel{\angle c}$$

$$= \pi - \pi$$

$$= 0$$

2. (25 pts.) For this problem, you should use the formulas provided in the handout as much as possible.

Consider the 6-point signal $x[n] = \cos(\pi n/3),\ n = 0,\dots,5$.

a. (8) Find and sketch the 6-point DFT $X[k]$ of this signal.

Now consider the 6-point signal $y[n] = \cos(\pi n/6),\ n = 0,\dots,5$.

b. (15) Find and sketch the 6-point DFT $Y[k]$ of this signal.

c. (2) Explain the relationship between $X[k]$ and $Y[k]$.

(a) $x[n] = \cos(\pi n/3) \qquad n = 0,\dots,5$

$$= \frac{1}{2}\left[e^{+j\frac{\pi n}{3}} + e^{-j\frac{\pi n}{3}}\right] \qquad n = 0,\dots,5$$

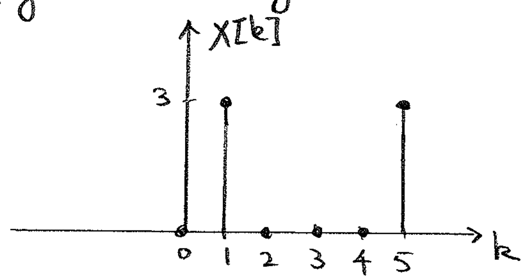
$$= \frac{1}{2}\left[e^{+j\frac{2\pi n}{6}} + e^{-j\frac{2\pi n}{6}}\right] \qquad n = 0,\dots,5$$

using transform relation:

$$\cos(2\pi k_0 n/N),\ 0 \le k \le N-1 \longleftrightarrow \frac{N}{2}\{\delta(k-k_0) + \delta(k-(N-k_0))\} \quad 0 \le k \le N-1$$

for this case: $k_0 = 1,\ N = 6$

$$X[k] = 3\{\delta(k-1) + \delta(k-5)\}, \quad 0 \le k \le 5$$



(b) $y[n] = \cos(\pi n/6),\ n = 0,\dots,5$

$$= \frac{1}{2}\{e^{j\pi n/6} + e^{-j\pi n/6}\} \qquad n = 0,\dots,5$$

using transform relation:

$$e^{j\omega_0 n},\ n = 0,\dots,N-1 \longleftrightarrow \frac{\sin\left(\left(\frac{2\pi k}{N} - \omega_0\right)\left(\frac{N}{2}\right)\right)}{\sin\left(\left(\frac{2\pi k}{N} - \omega_0\right)\left(\frac{1}{2}\right)\right)} e^{-j\left(\frac{2\pi k}{N} - \omega_0\right)\left(\frac{N-1}{2}\right)}$$

2. (continued)

(b) continued

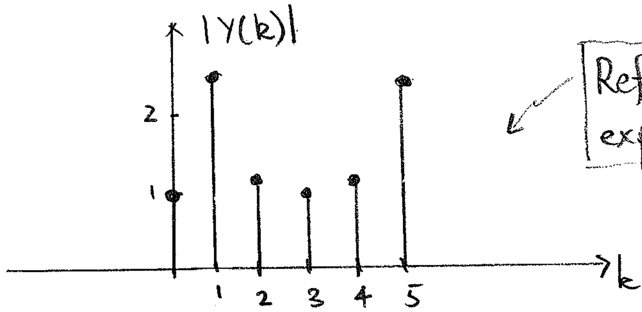
for this case : $\omega_0 = \frac{\pi}{6}$, $N=6$

$$e^{j\frac{\pi}{6}n},\ n=0,\dots,N-1 \Leftrightarrow \frac{\sin\left(\left(\frac{2\pi k}{6}-\frac{\pi}{6}\right)3\right)}{\sin\left(\left(\frac{2\pi k}{6}-\frac{\pi}{6}\right)\left(\frac{1}{2}\right)\right)} e^{-j\left(\frac{2\pi k}{6}-\frac{\pi}{6}\right)\left(\frac{5}{2}\right)}$$

$$= \frac{\sin\left(\frac{2\pi}{6}\left(k-\frac{1}{2}\right)3\right)}{\sin\left(\frac{2\pi}{6}\left(k-\frac{1}{2}\right)\frac{1}{2}\right)} e^{-j\frac{2\pi}{6}\left(k-\frac{1}{2}\right)\left(\frac{5}{2}\right)}$$

$$= \frac{\sin\left(\pi\left(k-\frac{1}{2}\right)\right)}{\sin\left(\frac{\pi}{6}\left(k-\frac{1}{2}\right)\right)} e^{-j\frac{5\pi}{6}\left(k-\frac{1}{2}\right)}$$

$$\therefore\ Y(k) = \frac{1}{2}\left[\frac{\sin\left(\pi\left(k-\frac{1}{2}\right)\right)}{\sin\left(\frac{\pi}{6}\left(k-\frac{1}{2}\right)\right)} e^{-j\frac{5\pi}{6}\left(k-\frac{1}{2}\right)} + \frac{\sin\left(\pi\left(k+\frac{1}{2}\right)\right)}{\sin\left(\frac{\pi}{6}\left(k+\frac{1}{2}\right)\right)} e^{-j\frac{5\pi}{6}\left(k+\frac{1}{2}\right)}\right]$$



Refer to next page for explanation of plot

(c) Leakage occurs for $Y(k)$, whereas no leakage occurs for $X(k)$. Both $Y(k)$ and $X(k)$ are DFT's of 6-pt windowed cosines.

2.(b) Clarification

let $y[n] = \cos(\omega_0 n)$, $n = 0, \dots, N-1$

$$Y(k) = \frac{1}{2}\left\{ \underbrace{\frac{\sin\left(\left(\frac{2\pi k}{N} - \omega_0\right)\frac{N}{2}\right)}{\sin\left(\left(\frac{2\pi k}{N} - \omega_0\right)\frac{1}{2}\right)} e^{-j\left(\frac{2\pi k}{N} - \omega_0\right)\left(\frac{N-1}{2}\right)}}_{psinc_1} + \underbrace{\frac{\sin\left(\left(\frac{2\pi k}{N} + \omega_0\right)\frac{N}{2}\right)}{\sin\left(\left(\frac{2\pi k}{N} + \omega_0\right)\frac{1}{2}\right)} e^{-j\left(\frac{2\pi k}{N} + \omega_0\right)\frac{N-1}{2}}}_{psinc_2} \right\}$$

For N large, $psinc_1$ and $psinc_2$ has minimal overlap
$\Rightarrow$ can approximately sketch as one psinc located at ω_0 and another located at $-\omega_0$.

<u>But</u>

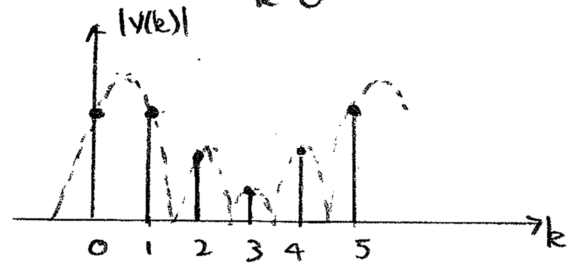
For N small (such as the case here with $N = 6$), $psinc_1$ and $psinc_2$ has <u>significant overlap</u>. The values for $k = 0, \dots, 5$ are listed below for $psinc_1$ and $psinc_2$, $\omega_0 = \pi/6$.

k	$\frac{1}{2}psinc_1(k)$	$\frac{1}{2}psinc_2(k)$	$\frac{1}{2}\{psinc_1 + psinc_2\} = Y(k)$
0	0.5+1.867i	0.5 − 1.867i	1
1	0.5 − 1.867i	0.5 − 0.5i	1 − 2.366i
2	0.5 − 0.5i	0.5 − 0.134i	1 − 0.634i
3	0.5 − 0.134i	0.5 + 0.134i	1
4	0.5 + 0.134i	0.5 + 0.5i	1 + 0.634i
5	0.5 + 0.5i	0.5 + 1.866i	1 + 2.366i

note the significant overlap at each k, also note that the values are complex

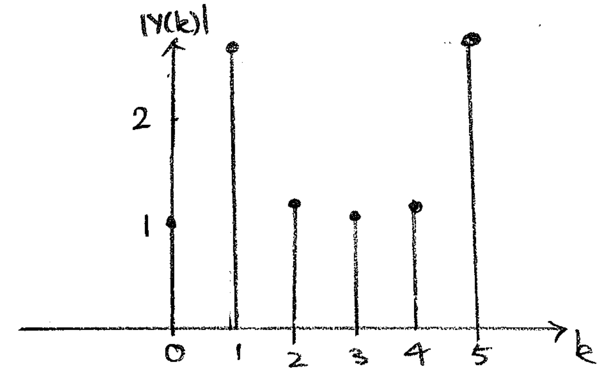
Key point
$|Y(k)| = \frac{1}{2}|psinc_1 + psinc_2| \not\approx \frac{1}{2}|psinc_1| + \frac{1}{2}|psinc_2|$ for N small

$\therefore$ $|Y(k)| = [\,1 \quad 2.569 \quad 1.184 \quad 1 \quad 1.184 \quad 2.569\,]$
($\uparrow$ $k=0$, $\uparrow$ $k=5$)



this is not a very accurate graph due to overlap between the two psinc's for N small

$\Rightarrow$



This is an accurate graph.

3. a. (21) Derive a decimation-in-time FFT algorithm for a 9-point DFT, and draw a complete flow diagram for the algorithm.

   b. (4) Calculate the approximate number of complex operations required to compute the 9-point FFT and compare with the number of complex operations required to compute the 9-point DFT directly.

(a) $9 = 3 \times 3$, so split the 9-pt DFT into 3 3-pt DFTs using decimation in time

$$X^{(9)}(k) = \sum_{n=0}^{8} x[n] e^{-j2\pi kn/9}$$

$$= \sum_{n=0}^{2} x[3n] e^{-j2\pi k(3n)/9} + \sum_{n=0}^{2} x[3n+1] e^{-j2\pi k(3n+1)/9} + \sum_{n=0}^{2} x[3n+2] e^{-j2\pi k(3n+2)/9}$$

$$= X_0^{(3)}(k) + e^{-j\frac{2\pi}{9}k} X_1^{(3)}(k) + e^{-j\frac{2\pi}{9}2k} X_2^{(3)}(k)$$

$$= X_0^{(3)}(k) + W_9^k X_1^{(3)}(k) + W_9^{2k} X_2^{(3)}(k)$$

For each 3-pt DFT

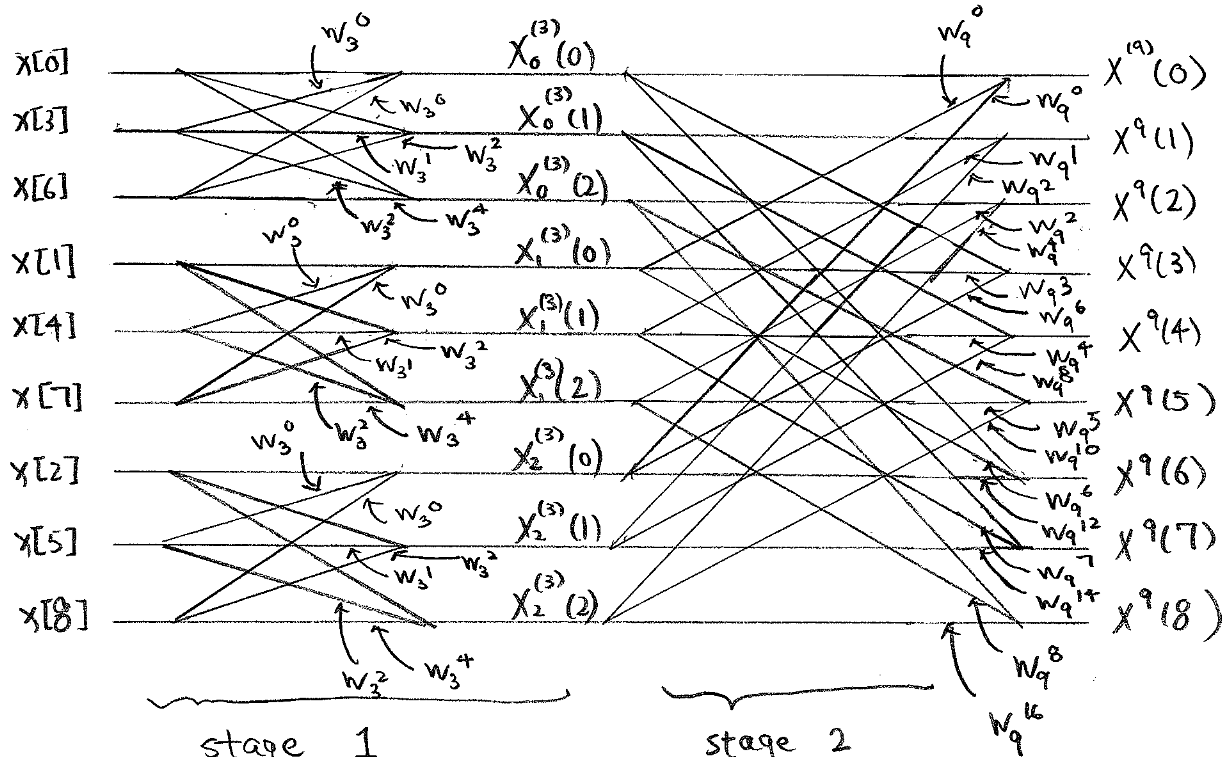
$$Y^{(3)}(k) = \sum_{n=0}^{2} y[n] e^{-j\frac{2\pi kn}{3}}$$

$$= y[0] + y[1] e^{-j\frac{2\pi k}{3}} + y[2] e^{-j\frac{4\pi k}{3}}$$

$$= y[0] + W_3^k y[1] + W_3^{2k} y[2]$$

↓ diagram on next page

3. (continued)



(b)

fft:

$3(3)(2) = 18$ multiplies + $9(2) = 18$ multiplies = 36 multiplies

$3(3)(2) = 18$ adds + $9(2) = 18$ adds = 36 adds

$\Rightarrow 18 + 18 + 18 + 18 = 72$ complex operations counting additions and multiplications

DFT:

N^2 multiplies = 81

$N(N-1) = N^2 - N$ adds = 72

153 complex operations counting additions and multiplications

4. (25 pts) Let θ be a random variable that is uniformly distributed on the interval $[0, 2\pi)$. Define a new random variable $X = \cos(2\pi f_0 t + \theta)$ for some fixed value of f_0 and a fixed value of t.

a) (8) Find the mean of X.

b) (8) Find the variance of X.

Now suppose that we define a second random variable $Y = \cos(2\pi f_0 s + \theta)$, where f_0 has the same value as before and s is a fixed parameter $\neq t$.

c) (9) Find the expected value of the product XY.

Note: The following formula may be of benefit to you in working this problem.

$$\cos\alpha\cos\beta = \tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta)$$

(a) $E[X] = E[\overbrace{\cos(2\pi f_0 t + \theta)}^{g(\theta)\text{: this is a function of } \theta}] = E[g(\theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta)\, d\theta$

$= \int_{-\infty}^{\infty} \cos(2\pi f_0 t + \theta) f_\Theta(\theta)\, d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi f_0 t + \theta)\, d\theta$ given that θ is uniformly distributed

$= \frac{1}{2\pi}\sin(2\pi f_0 t + \theta)\Big|_0^{2\pi}$

$= \frac{1}{2\pi}[\sin(2\pi f_0 t + 2\pi) - \sin(2\pi f_0 t)]$

$= \frac{1}{2\pi}[\sin(2\pi f_0 t) - \sin(2\pi f_0 t)]$

$= 0$

(b) $E[X^2] = E[\cos^2(2\pi f_0 t + \theta)]$

$= \int_{-\infty}^{\infty} \cos^2(2\pi f_0 t + \theta) f_\Theta(\theta)\, d\theta$ note: $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_0 t + 2\theta)\right) d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\, d\theta + \cancel{\int_0^{2\pi} \frac{1}{2}\cos(4\pi f_0 t + 2\theta)\, d\theta} \to 0$ as before

$= \frac{1}{2\cancel{\pi}}\cancel{\pi} = \frac{1}{2}$

$Var(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - 0^2$

$= \frac{1}{2}$

4. (continued)

(c) $Y = \cos(2\pi f_0 s + \theta)$

let $Z = XY = \cos(2\pi f_0 t + \theta)\cos(2\pi f_0 s + \theta)$

$$= \frac{1}{2}\cos(2\pi f_0 t + \theta + 2\pi f_0 s + \theta) + \frac{1}{2}\cos(2\pi f_0 t + \cancel{\theta} - 2\pi f_0 s - \cancel{\theta})$$

$$= \frac{1}{2}\cos(2\pi f_0(t+s) + 2\theta) + \frac{1}{2}\cos(2\pi f_0(t-s))$$

note that $Z = g(\theta)$: ie. Z is a function of $\theta \Rightarrow E[g(\theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta)\, d\theta$

$$\therefore \quad E[XY] = E\left[\frac{1}{2}\cos(2\pi f_0(t+s) + 2\theta) + \frac{1}{2}\cos(2\pi f_0(t-s))\right]$$

$$= \frac{1}{2}E\left[\cos(2\pi f_0(t+s) + 2\theta)\right] + \frac{1}{2}E\left[\underbrace{\cos(2\pi f_0(t-s))}_{\text{this is not a function of } \theta}\right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \cos(2\pi f_0(t+s) + 2\theta) f_\Theta(\theta)\, d\theta + \frac{1}{2}\cos(2\pi f_0(t-s))$$

$$= \underbrace{\frac{1}{2}\int_{0}^{2\pi} \cos(2\pi f_0(t+s) + 2\theta)\, d\theta}_{0 \text{ as before}} + \frac{1}{2}\cos(2\pi f_0(t-s))$$

$$= \frac{1}{2}\cos(2\pi f_0(t-s))$$

1.	______
2.	______
3.	______
4.	______
Total	______