13 February 2008

Name: Key

ECE 438 **Exam No. 1** **Spring 2008**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the system defined by the difference equation

$$y[n] = x[n] - x[n-1] - y[n-1].$$

a. (2) Is this system linear? Just state "yes" or "no". You do not need to justify your answer, and you will not receive any credit for a proof of your answer.

b. (2) Is this system time-invariant? Just state "yes" or "no". You do not need to justify your answer, and you will not receive any credit for a proof of your answer.

c. (5) Assuming that the system is initially at rest, i.e. $y[n] = 0, n < 0$, find the impulse response $h[n]$ of this system.

d. (10) Find the frequency response of this system $H(\omega)$ by directly computing the response to the input $x[n] = e^{j\omega n}$.

e. (6) Find simple expressions for the magnitude and phase of the frequency response of this system.

(a) Yes

(b) Yes

(c) let input $x[n] = \delta[n]$, then $y[n] = h[n]$

n	$y[n]$
-1	0
0	1
1	-2
2	2
3	-2
$\vdots$	$\vdots$

$$h[n] = \delta[n] + 2(-1)^n u[n-1]$$

1. (continued)

(d) let $x[n] = e^{j\omega n}$, assume $y[n] = H(\omega)e^{j\omega n}$

$$H(\omega)e^{j\omega n} = e^{j\omega n} - e^{j\omega(n-1)} - e^{j\omega(n-1)} H(\omega)$$

$$H(\omega)\cancel{e^{j\omega n}} = \cancel{e^{j\omega n}} - \cancel{e^{j\omega n}} e^{-j\omega} - \cancel{e^{j\omega n}} e^{-j\omega} H(\omega)$$

$$H(\omega)(1+e^{-j\omega}) = 1 - e^{-j\omega}$$

$$H(\omega) = \frac{1-e^{-j\omega}}{1+e^{-j\omega}}$$

(e) $$H(\omega) = \frac{\cancel{e^{-j\frac{\omega}{2}}}(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})}{\cancel{e^{-j\frac{\omega}{2}}}(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}})} \cdot \frac{\frac{1}{2j}}{\frac{1}{2j}}$$

$$= \frac{\sin(\frac{\omega}{2})}{\frac{1}{j}\cos(\frac{\omega}{2})}$$

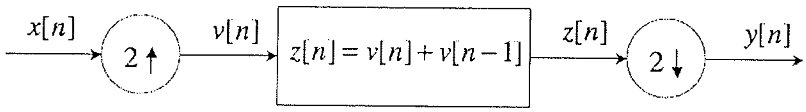
$$= \frac{j\sin(\frac{\omega}{2})}{\cos(\frac{\omega}{2})}$$

$$= j\tan(\frac{\omega}{2})$$

$$|H(\omega)| = |\tan(\frac{\omega}{2})|$$

$$\angle H(\omega) = \begin{cases} \frac{\pi}{2} & \text{for } \tan(\frac{\omega}{2}) \geq 0 \\ \frac{\pi}{2} \pm \pi & \text{for } \tan(\frac{\omega}{2}) < 0 \end{cases}$$

2. (25 pts.) Consider the system shown in the block diagram below.



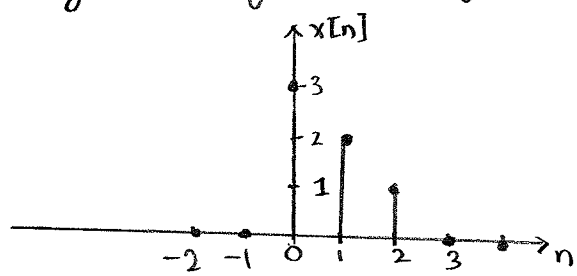
a. (10) For the signal $x[n]=(3-n)\big(u[n]-u[n-3]\big)$, sketch $x[n]$, $v[n]$, $z[n]$, and $y[n]$.

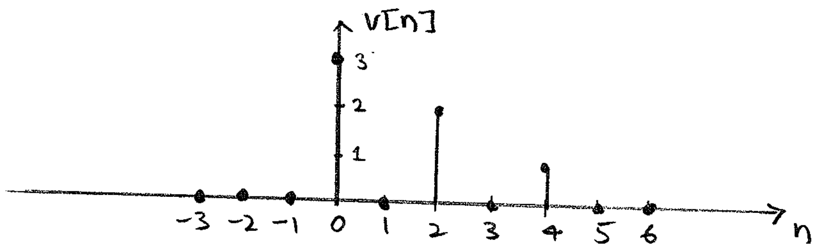
b. (15) Using transform relations, express the entire system above in the DTFT domain, i.e. find expressions for $V(\omega)$ in terms of $X(\omega)$, for $Z(\omega)$ in terms of $V(\omega)$, for $Y(\omega)$ in terms of $Z(\omega)$, and finally for $Y(\omega)$ in terms of $X(\omega)$. Each expression should be simplified as much as possible.

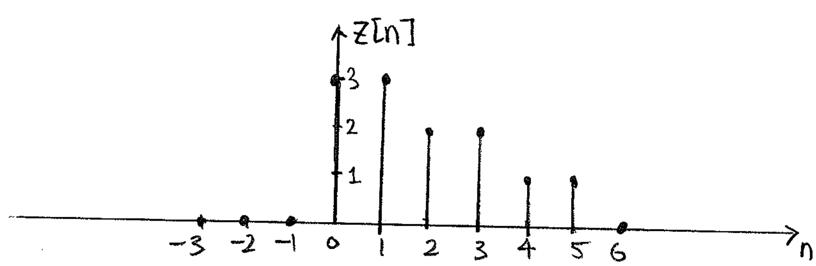
(a) $x[n] = (3-n)\big(u[n] - u[n-3]\big)$

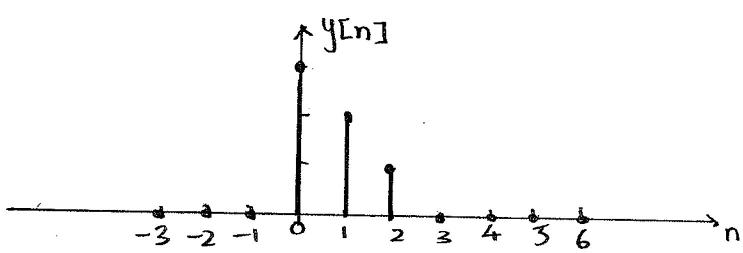
$= (3-n)\big(\delta[n] + \delta[n-1] + \delta[n-2]\big)$

$= 3\delta[n] + 2\delta[n-1] + \delta[n-2]$









2. (continued)

(b) $V(\omega) = X(2\omega)$

$$Z(\omega) = V(\omega) + e^{-j\omega} V(\omega)$$
$$= V(\omega)(1+e^{-j\omega})$$
$$= X(2\omega)(1+e^{-j\omega})$$

$$Y(\omega) = \frac{1}{2}\sum_{k=0}^{1} Z\left(\frac{\omega-2\pi k}{2}\right)$$
$$= \frac{1}{2}\sum_{k=0}^{1} \underbrace{X\left(\cancel{2}\left(\frac{\omega-2\pi k}{\cancel{2}}\right)\right)}_{X(\omega-2\pi k)}\left(1+e^{-j\left(\frac{\omega-2\pi k}{2}\right)}\right)$$
$$= \frac{1}{2}\left[X(\omega)(1+e^{-j\frac{\omega}{2}}) + \underbrace{X(\omega-2\pi)}_{}\left(1+e^{-j\left(\frac{\omega-2\pi}{2}\right)}\right)\right]$$

$X(\omega-2\pi) = X(\omega)$
since DTFT is repetitive
with period 2π

$$= \frac{1}{2}\left[X(\omega) + e^{-j\frac{\omega}{2}}X(\omega) + X(\omega) + e^{-j\frac{\omega}{2}}\underbrace{e^{+j\pi}}_{-1}X(\omega)\right]$$

$$= \frac{1}{2}\left[X(\omega) + X(\omega) + \cancel{e^{-j\frac{\omega}{2}}X(\omega)} - \cancel{e^{-j\frac{\omega}{2}}X(\omega)}\right]$$

$$\boxed{Y(\omega) = X(\omega)}$$

note: recall $y[n] = x[n]$ from part (a)

3. (25) Consider the two signals

$$x[n] = 2^{-(n-5)}u[n-5]$$
$$y[n] = u[n+4] - u[n-5]$$

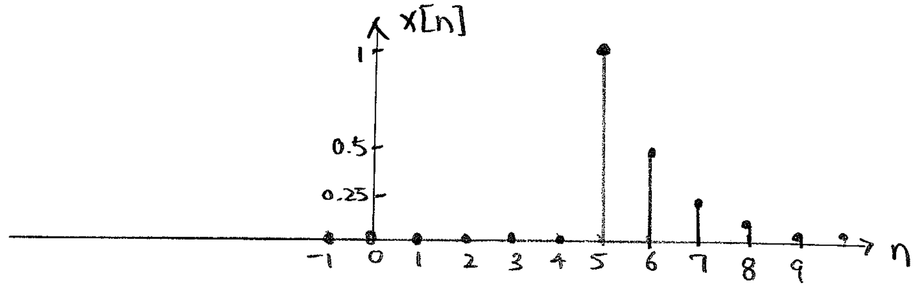
a. (4) Carefully sketch each of these signals.

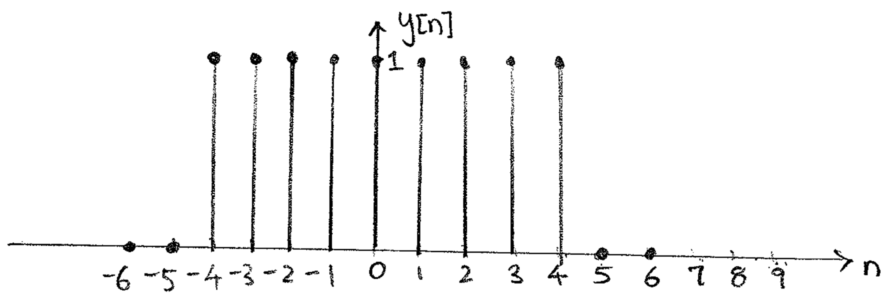
b. (3) Find the energy E_x for $x[n]$.

c. (2) Find the average value $\cancel{A_y}$ avg_y for $y[n]$.

d. (16) Compute the convolution $z[n] = x[n] * y[n]$ for these two signals, and sketch your result.

(a)





(b) $$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=5}^{\infty} \left|\left(\tfrac{1}{2}\right)^{n-5}\right|^2 = \sum_{n=5}^{\infty} \left(\tfrac{1}{2}\right)^{2(n-5)} = \sum_{n=5}^{\infty} \left(\tfrac{1}{4}\right)^{n-5} \quad \text{let } m = n-5$$

$$= \sum_{m=0}^{\infty} \left(\tfrac{1}{4}\right)^{m} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$\boxed{E_x = 4/3}$$

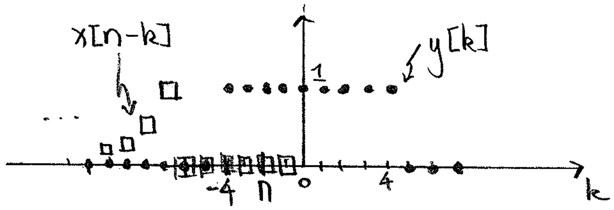
$$y_{avg} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cdot (9)$$

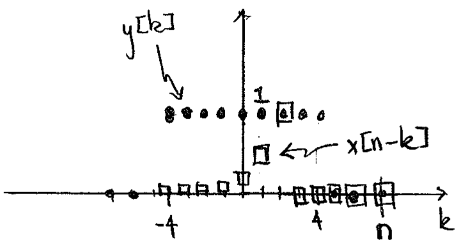
(c) $$\boxed{y_{avg} = 0}$$

3. (continued)

(d) <u>case 1</u>: $n \le 0 \Rightarrow z[n] = 0$



<u>case 2</u>: $0 < n \le 9$



$$z[n] = x[n] * y[n]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, y[k]$$

$$= \sum_{k=-4}^{4} x[n-k]$$

$$= \sum_{k=-4}^{n-5} 2^{-(n-k-5)}$$

$$= 2^{-(n-5)} \sum_{k=-4}^{n-5} 2^{k}$$

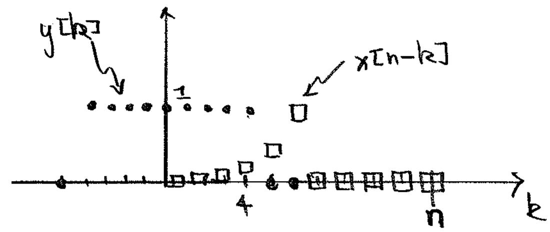
$$= 2^{-(n-5)} \frac{2^{-4} - 2^{n-4}}{1-2}$$

$$= 2^{-(n-5)}\, 2^{-4}(2^{n}-1)$$

$$= 2^{-n} \cdot 2 \cdot (2^{n}-1)$$

$$= 2 - 2^{-n+1}$$

<u>case 3</u>: $n > 9$



$$z[n] = \sum_{k=-4}^{4} x[n-k]$$

$$= \sum_{k=-4}^{4} 2^{-(n-k-5)}$$

$$= 2^{-(n-5)} \sum_{k=-4}^{4} 2^{k}$$

$$= 2^{-(n-5)} \frac{2^{-4} - 2^{5}}{1-2}$$

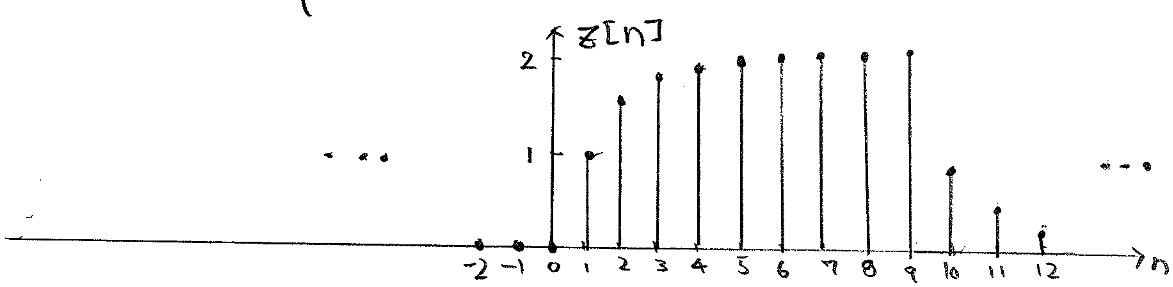
$$= 2^{-(n-5)} (2^{5} - 2^{-4})$$

$$= 2^{-n+10} - 2^{-n+1}$$

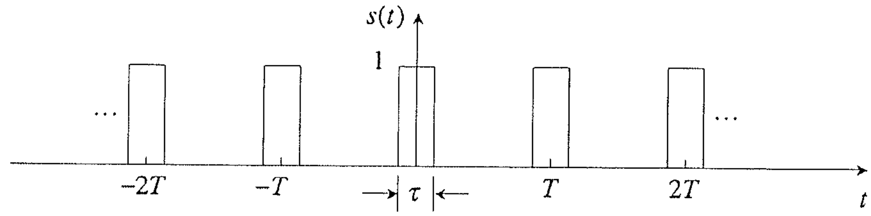
$$= 2^{-n}(2^{10} - 2)$$

$$= (1022)\, 2^{-n}$$

$$\therefore\ z[n] = \begin{cases} 0, & n \le 0 \\ 2 - 2^{-n+1}, & 0 < n \le 9 \\ (1022)\, 2^{-n} & n > 9 \end{cases}$$



4. (25 pts) Consider the waveform $s(t)$ shown below.



a) (5) Using the replication operator and the rect function, write a compact expression for $s(t)$.

b) (7) Find a simple expression for the CTFT $S(f)$ of $s(t)$ based on your answer to part (a) above. Your final answer should not include any operators, such as rep or comb.

c) (3) Carefully sketch $S(f)$, being sure to show all dimensions of its features in terms of the parameters τ and T.

Suppose that we use $s(t)$ to generated a sampled version $x_s(t)$ of an arbitrary signal $x(t)$ according to

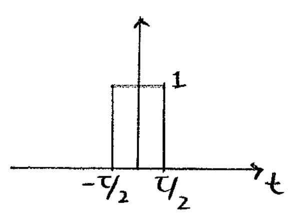
$$x_s(t) = s(t)x(t)$$

d) (2) Draw a "typical" waveform $x(t)$, and sketch what $x_s(t)$ would look like.

e) (5) Using transform relations, find a simple expression for the CTFT $X_s(f)$ of $x_s(t)$ in terms of the CTFT $X(f)$ for $x(t)$. Your final answer should not include any operators, such as rep, comb, or convolution.

f) (3) Assuming that $X(f) = (1 - |f|/W)\text{rect}(f/(2W))$, where $W < 1/(2T)$, sketch $X_s(f)$.

(a)



$$\text{rect}(t/\tau) = \begin{cases} 1 & |t| < \frac{\tau}{2} \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow s(t) = \text{rep}_T[\text{rect}(t/\tau)]$$

(b) $$S(f) = \frac{1}{T}\text{comb}_{\frac{1}{T}}[\tau\,\text{sinc}(\tau f)]$$

$$= \frac{\tau}{T}\sum_{k=-\infty}^{\infty} \text{sinc}\left(\tau\frac{k}{T}\right)\delta\left(f - \frac{k}{T}\right)$$

4. (continued)

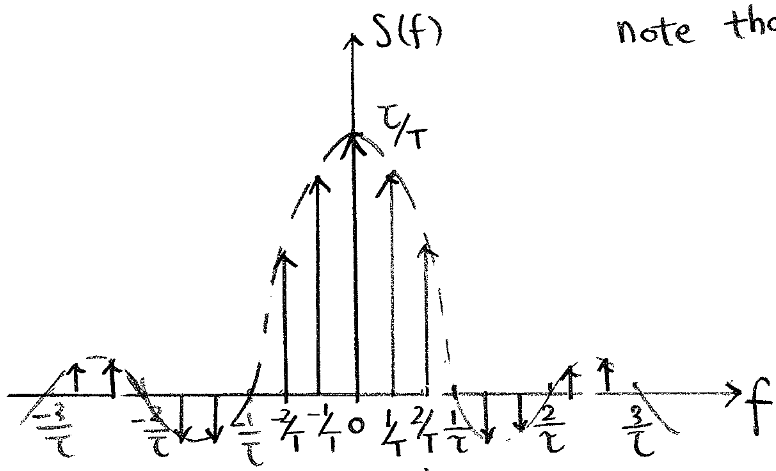
(c) $\mathrm{sinc}(\tau f) = \dfrac{\sin \pi \tau f}{\pi \tau f}$

zero-crossings: $\cancel{\pi} \tau f = m \cancel{\pi}$, $m = \pm 1, \pm 2, \ldots$

$$f = \frac{m}{\tau}$$

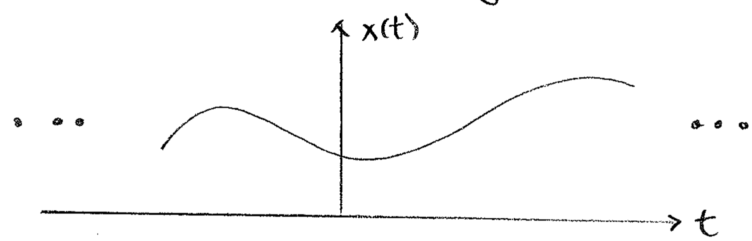
note that $\tau < T$

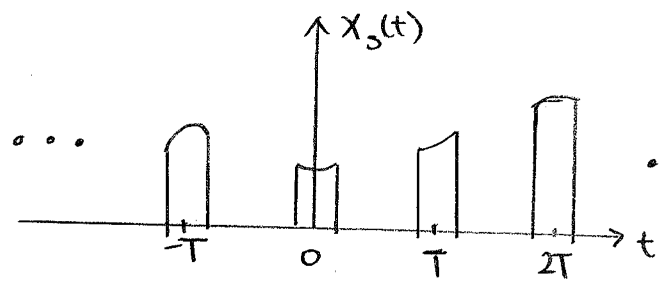
$\therefore \frac{1}{T} < \frac{1}{\tau}$



spacing between impulses depend on T

(d)





(e) $x_s(t) = s(t)\, x(t)$

$$X_s(f) = S(f) * X(f)$$

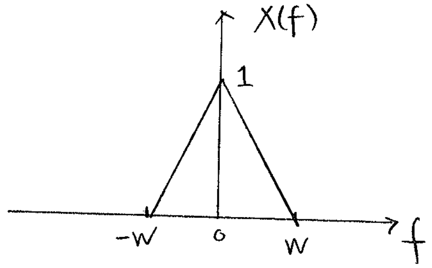
$$= \left[\frac{\tau}{T} \sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\tau \frac{k}{T}\right) \delta\left(f - \frac{k}{T}\right) \right] * X(f)$$

since convolution is linear

$$= \frac{\tau}{T} \sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\tau \frac{k}{T}\right) \left[\delta\left(f - \frac{k}{T}\right) * X(f) \right] = \frac{\tau}{T} \sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\tau \frac{k}{T}\right) X\left(f - \frac{k}{T}\right)$$

4. (continued)

(f) $X(f) = (1 - |f|/W)\, \mathrm{rect}(f/2W)$



given $W < \frac{1}{2T}$