3 May 2007 **Name:** solution

ECE 438 **Final Exam** **Spring 2007**

- You have 120 minutes to work the following five problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the causal digital filter described by the following system equation

$$y[n] = x[n] - y[n-1]$$

a. (5) Find the response of this system for times $n \geq 0$ to the input $x[n] = u[n]$, assuming that $y[-1] = 0.5$.

b. (5) Find a simple expression for the frequency response $H(\omega)$ of this system.

c. (5) Sketch the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response.

d. (5) Find the impulse response $h[n]$ for this system.

e. (5) Is this a stable system? Justify your answer.

a. $y[0] = x[0] - y[-1] = 0.5$

$y[1] = x[1] - y[0] = 0.5$

$\vdots \quad \vdots \quad \vdots$

$y[n] = x[n] - y[n-1] = 0.5$

so $y[n] = 0.5$ for $n \geq 0$.

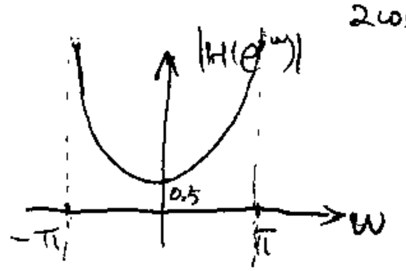
b. $Y(e^{j\omega})(1 + e^{-j\omega}) = X(e^{j\omega})$

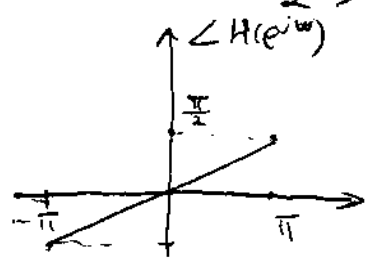
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1+e^{-j\omega}}$$

c.
$$H(e^{j\omega}) = \frac{1}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)} = \frac{1}{e^{-j\frac{\omega}{2}} \cdot 2\cos\frac{\omega}{2}} = \frac{e^{j\frac{\omega}{2}}}{2\cos\frac{\omega}{2}}$$

$$|H(e^{j\omega})| = \frac{1}{2\left|\cos\frac{\omega}{2}\right|} = \frac{1}{2\cos\frac{\omega}{2}} \qquad \angle H(e^{j\omega}) = \frac{\omega}{2}$$

since $\cos\frac{\omega}{2} \geq 0$ for $-\pi \leq \omega \leq \pi$.





1. (continued)

d. The inverse DTFT of $H(e^{j\omega}) = \frac{1}{1+e^{-j\omega}}$

is $h[n] = (-1)^n u[n]$

e. The system is not stable, since the pole -1 is on the unit circle.

2. (25 pts.) Suppose that you wish to build a hardware system that will compute the exact DFT of a length 1280-point signal. You have at your disposal a chip that will compute a length 256-point FFT.

   a. (12) Derive a set of equations that show exactly how to efficiently compute the 1280-point DFT using one or more of the 256-point FFT chips.

   b. (7) Draw a block diagram for your system being careful to show all relevant details.

   c. (3) Determine how many complex operations would be required to evaluate the 1280-point DFT directly.

   d. (3) Determine how many complex operations would be required to evaluate the 1280-point DFT using your system with the 256-point FFT chips.

a. $N = 1280, \quad \frac{N}{5} = 256$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{m=0}^{N/5-1} x[5m] e^{-j\frac{2\pi}{N}k\cdot 5m} + \sum_{m=0}^{\frac{N}{5}-1} x[5m+1] e^{-j\frac{2\pi}{N}k(5m+1)}$$

$$+ \sum_{m=0}^{\frac{N}{5}-1} x[5m+2] e^{-j\frac{2\pi}{N}k(5m+2)} + \sum_{m=0}^{\frac{N}{5}-1} x[5m+3] e^{-j\frac{2\pi}{N}k(5m+3)}$$

$$+ \sum_{m=0}^{\frac{N}{5}-1} x[5m+4] e^{-j\frac{2\pi}{N}k(5m+4)}$$

Let the $\frac{N}{5}$ point DFT of $\{x[5m+i]\}_{0 \le m < \frac{N}{5}}$ be $X_i[k]$,

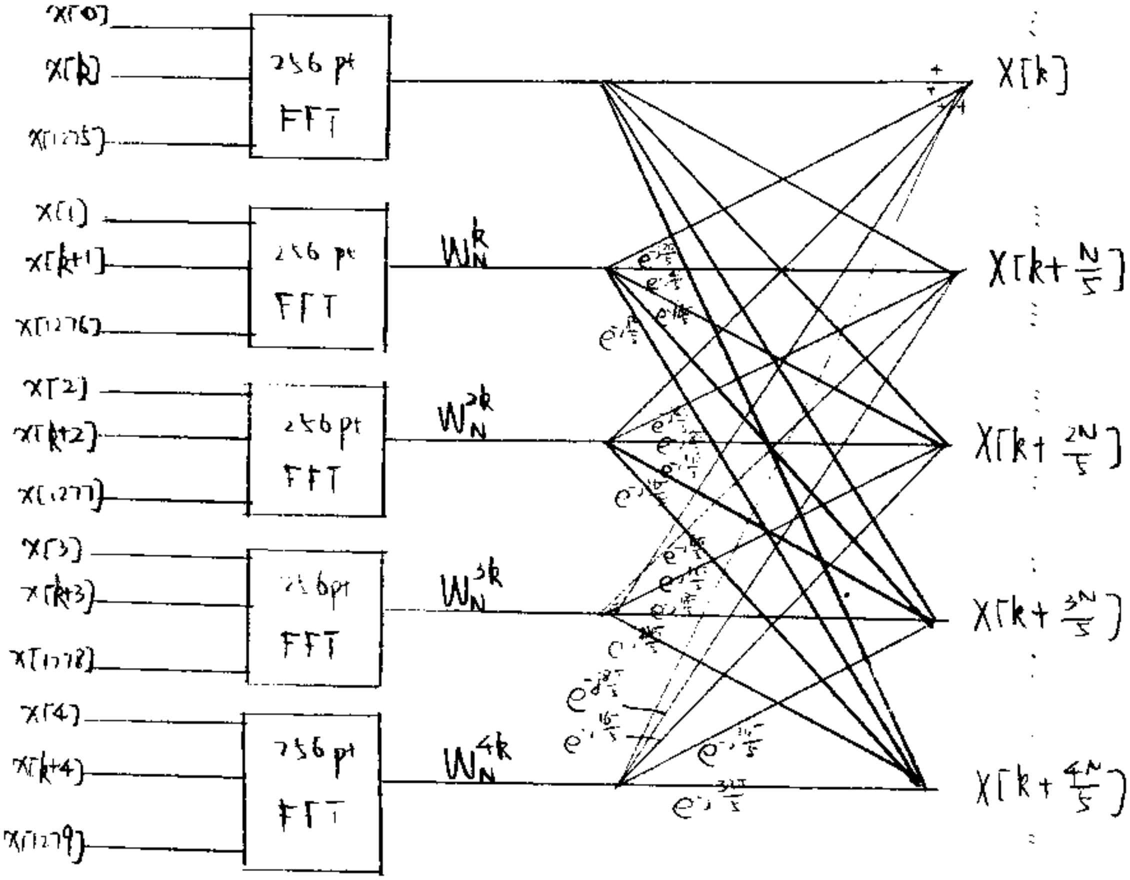
where $i = 0, 1, 2, 3, 4$.

Then
$$X[k + \tfrac{N}{5} j] = X_0[k] + e^{-j\frac{2\pi}{N}k} e^{-j\frac{2\pi}{5}j} X_1[k]$$
$$+ e^{-j\frac{2\pi}{N}\cdot 2k} e^{-j\frac{2\pi}{5}j\cdot 2} X_2[k]$$
$$+ e^{-j\frac{2\pi}{N}3k} e^{-j\frac{2\pi}{5}j\cdot 3} X_3[k]$$
$$+ e^{-j\frac{2\pi}{N}4k} e^{-j\frac{2\pi}{5}j\cdot 4} X_4[k]$$

$(0 \le k < 256, \quad j = 0, 1, 2, 3, 4)$.

alternatively: $X[k] = X_0[k] + W_N^{k} X_1[k] + W_N^{2k} X_2[k] + W_N^{3k} X_3[k] + W_N^{4k} X_4[k]$

2 b)


x[0]
x[k]
x[1275]
256 pt FFT
x[1]
x[k+1]
x[1276]
256 pt FFT
x[2]
x[k+2]
x[1277]
256 pt FFT
x[3]
x[k+3]
x[1278]
256 pt FFT
x[4]
x[k+4]
x[1279]
256 pt FFT
W_N^k
W_N^{2k}
W_N^{3k}
W_N^{4k}
X[k]
$X[k+\frac{N}{5}]$
$X[k+\frac{2N}{5}]$
$X[k+\frac{3N}{5}]$
$X[k+\frac{4N}{5}]$

2 c). N^2 complex multiplications + N^2-N complex additions

so $2N^2-N = 3275520$ complex operations.

Alternatively (approximately) $N^2 = 1638400$ complex operations.

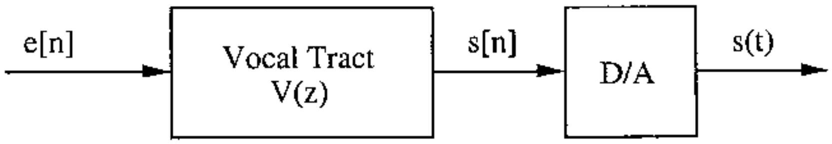
d) 256 point FFT requires $\frac{3}{2} \times 256 \log_2 256 = 3072$ complex operations.

Then $4N$ complex multiplications and $4N$ additions,

so $3072 \times 5 + 4N + 4N = 25600$ complex operations

Or approximately $5 \times 256 \log_2 256 + 4N = 15360$ complex operations.

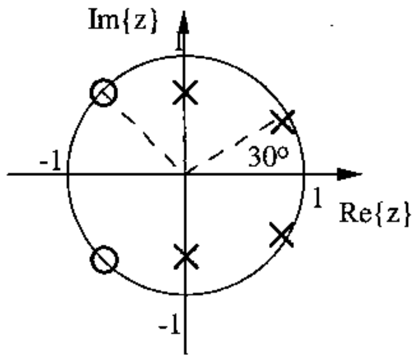
3. (25) The digital synthesizer for voiced speech shown below operates at an 8 kHz sampling rate.



The excitation is given by

$$e[n] = \sum_{k=-\infty}^{\infty} \delta[n-40k]$$

The vocal tract transfer function V(z) has poles and zeros at the locations shown below:



a. (5) What is the pitch period in seconds?

b. (6) Find the formant frequencies in Hz, and rank them according to their strength, *i.e.* how peaked the vocal tract response is at the corresponding frequency.

c. (7) Sketch what a *wideband* spectrogram would look like for this utterance. Be sure to label the pitch and formant information appropriately.

d. (7) Sketch what a *narrowband* spectrogram would look like for this utterance. Be sure to label the pitch and formant information appropriately.

a) $T_s = \frac{1}{8000}$ $\quad P = \frac{40}{8000} = \frac{1}{200} = 0.005$ sec. $= 5$ msec

$\frac{1}{P} = 200$ Hz

b) $\omega_d = \frac{2\pi f_a}{f_s} \Rightarrow f_a = \frac{\omega_d}{2\pi} f_s = \frac{\omega_d}{2\pi} 8000$

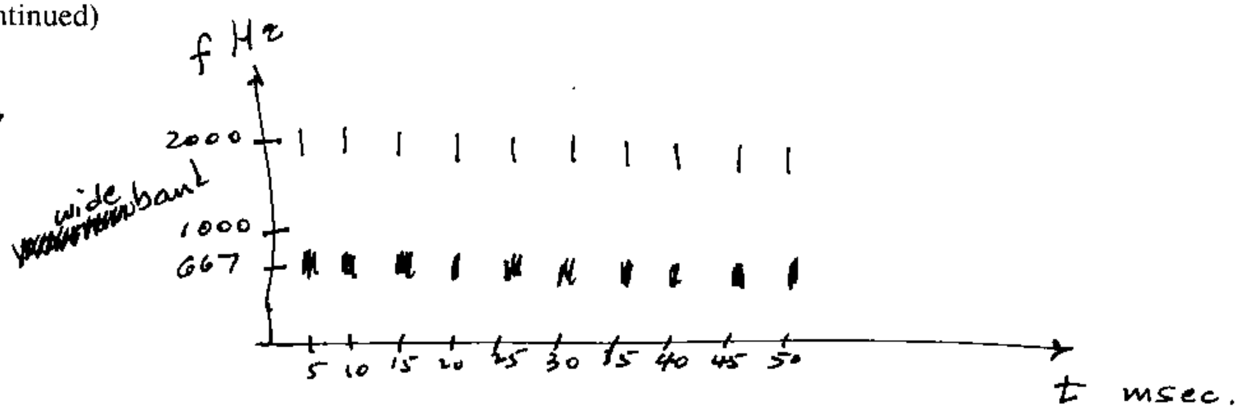
$\omega_{d_1} = \pi/6 \Rightarrow f_{a_1} = \frac{8000}{12} = 666.7$ Hz ← { strongest since poles are closest to unit circle

$\omega_{d_2} = \pi/2 \Rightarrow f_{a_2} = \frac{8000}{4} = 2000$ Hz

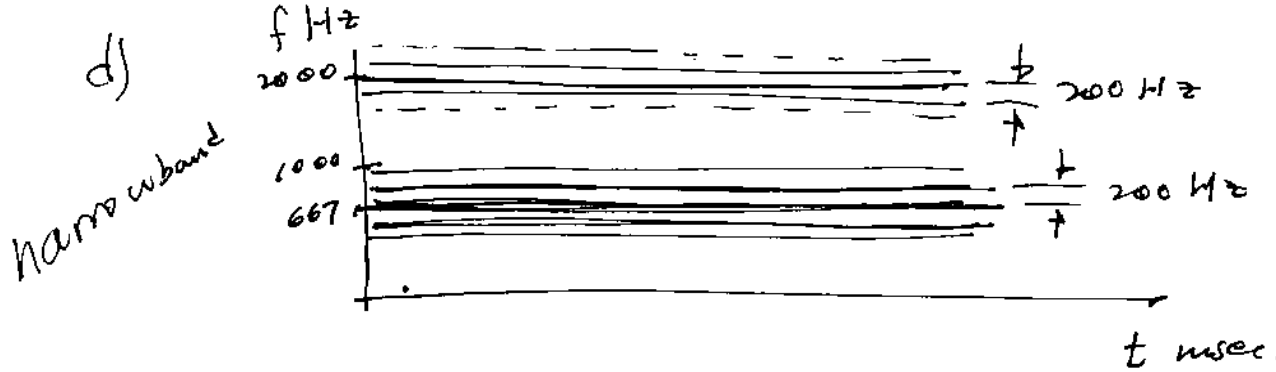
zero at $\omega_{d_3} = 3\pi/4$ doesn't contribute a formant.

3. (continued)

c)



d)



4. (25 pts) The three parts a), b), and c) below show three different continuous-space 2-D signals $f_a(x,y)$, $f_b(x,y)$, $f_c(x,y)$, respectively. For each signal, the shaded (cross-hatched) areas have value 1 and the non-shaded (white) areas have value 0.

For **each** part a), b), and c) below, do the following:

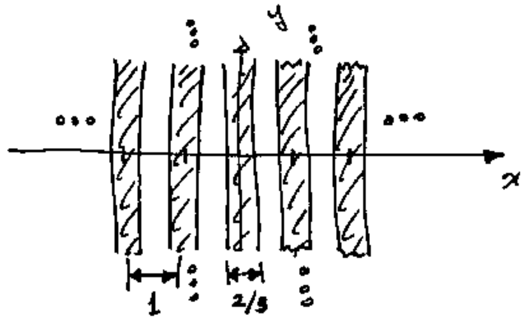
i) Express the 2-D signal ($f_a(x,y)$, $f_b(x,y)$, and $f_c(x,y)$) in terms of standard operators and special signals.

ii) Find the 2-D continuous-space Fourier transform (CSFT) ($F_a(u,v)$, $F_b(u,v)$, and $F_c(u,v)$) using standard transform relations and transform pairs for special signals.

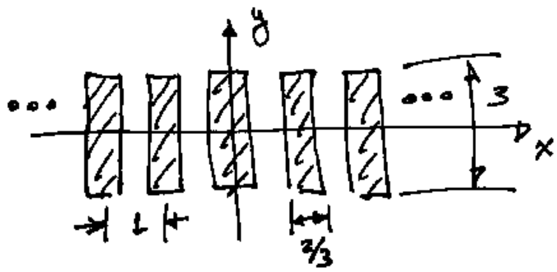
iii) Sketch the 2-D continuous-space Fourier transform (CSFT) ($F_a(u,v)$, $F_b(u,v)$, and $F_c(u,v)$). Be sure to dimension your drawing completely. The goal is convey the fact that you understand what the CSFT looks like.

Hint: It will be most efficient to use your answer to part a) as the starting point for deriving your answer to part b), and to use your answer to part b) as the starting point for deriving your answer to part c).

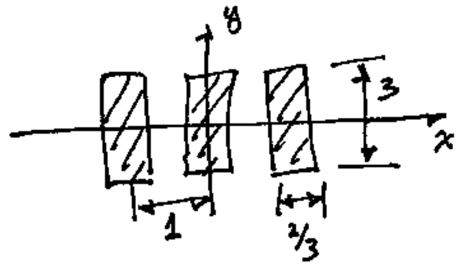
a) (8) The 2-D signal $f_a(x,y)$ consists of infinitely many vertical bars, each of infinite length. Each bar has width 2/3. The bars are located on centers separated by unit distance (1).



b) (8) The 2-D signal $f_b(x,y)$ consists of infinitely many vertical bars, each of length 3. Each bar has width 2/3. The bars are located on centers separated by unit distance (1).

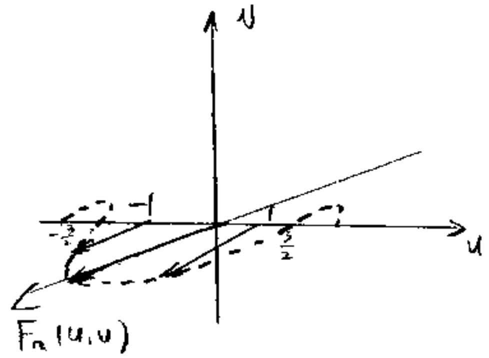


c) (9) The 2-D signal $f_c(x,y)$ consists of exactly 3 vertical bars, each of length 3. Each bar has width 2/3. The bars are located on centers separated by unit distance (1).



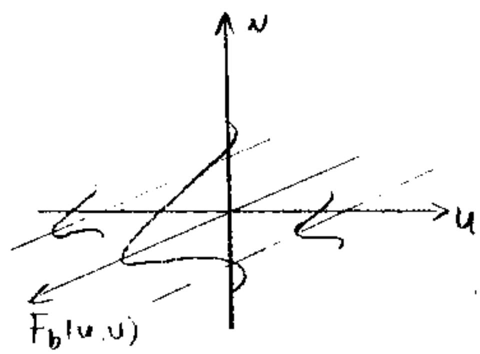
4. a) $f_a(x,y) = \mathrm{rep}_1\left(\mathrm{rect}\left(\frac{3}{2}x\right)\right)\cdot 1$

$F_a(u,v) = \mathrm{comb}_1\left[\frac{2}{3}\,\mathrm{sinc}\left(\frac{2}{3}u\right)\right]\delta(v)$



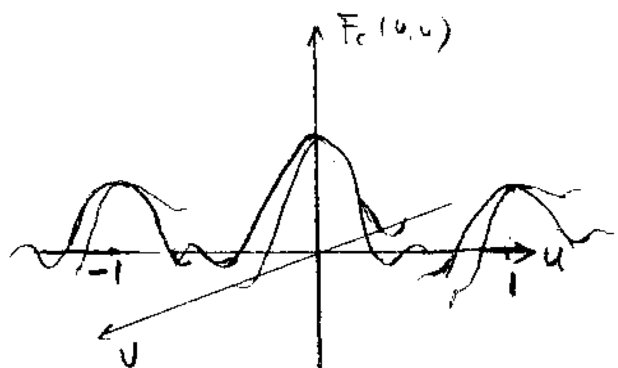
b) $f_b(x,y) = \mathrm{rep}_1\left(\mathrm{rect}\left(\frac{3}{2}x\right)\right)\mathrm{rect}\left(\frac{1}{3}y\right)$

$F_b(u,v) = 3\,\mathrm{comb}_1\left[\frac{2}{3}\mathrm{sinc}\left(\frac{2}{3}u\right)\right]\mathrm{sinc}(3v)$



c) $f_c(x,y) = f_a(x,y)\,\mathrm{rect}\left(\frac{1}{3}x\right)\mathrm{rect}\left(\frac{1}{3}y\right)$

$F_c(u,v) = F_a(u,v) ** \quad 9\,\mathrm{sinc}(3u)\,\mathrm{sinc}(3v)$



5. (25 pts) Consider a spatial filter with point spread function $h[m,n]$ given below

$$h[m,n] \qquad \begin{array}{cc|ccc} & & & n & \\ & & -1 & 0 & 1 \\ \hline & -1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ m & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}$$

a. (10) Find the output $g[m,n]$ when this filter is applied to the following input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 9x9 set of pixels in the input image.

$$\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

b. (12) Find a simple expression for the magnitude $|H(\mu,\nu)|$ of the frequency response of this filter, and sketch it along the μ and ν axes, and the $\mu=\nu$ axis.

c. (3) Compare your results from parts a and b, and explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's.

5 a.

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

b.

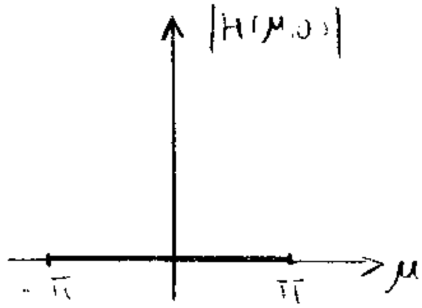
$$h[m,n] = \left(\delta[m+1] + \delta[m] + \delta[m-1]\right) \cdot \left(-\tfrac{1}{2}\delta[n+1] + \delta[n] - \tfrac{1}{2}\delta[n-1]\right)$$

$$H(\mu, \nu) = \left(e^{j\mu} + 1 + e^{-j\mu}\right)\left(-\tfrac{1}{2}e^{j\nu} + 1 - \tfrac{1}{2}e^{-j\nu}\right)$$

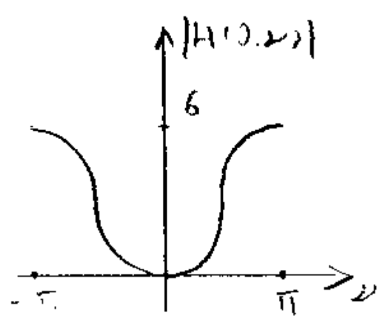
$$= (1 + 2\cos\mu)(1 - \cos\nu)$$

$$|H(\mu, \nu)| = |(1 + 2\cos\mu)(1 - \cos\nu)|$$

$|H(\mu, 0)| = 0$

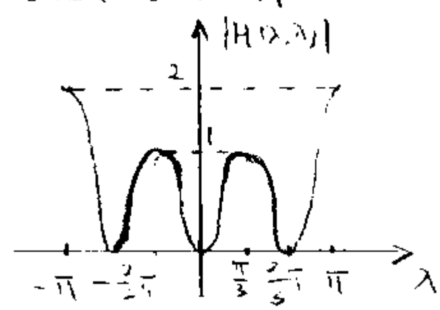


$|H(0,\nu)| = 3(1 - \cos\nu)$



$|H(\lambda,\lambda)| = |(1+2\cos\lambda)(1-\cos\lambda)|$

where $\mu = \nu = \lambda$



5 c. This filter detects vertical edges in the image. From the magnitude plots in part b, we know that it is high pass along the ν direction and “no pass” along the μ direction. So at the edge of the region of all 1’s, the output has opposite signs in neighboring columns, which means a vertical edge. The DC response of this filter is 0, so in the center of the region of all 1’s the output is 0.