18 April 2007 **Name:** Solution

ECE 438 **Exam No. 3** **Spring 2007**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider a sequence of random variables $X[n]$ that are independent and identically distributed with a mean of 1 and variance of 1.

   a. (5) Find the autocorrelation function $r_{XX}[n]$ for this signal.

   Suppose we define a new random process $Y[n]$ in terms of the process $X[n]$ defined above, according to

$$Y[n] = X[n] + X[n-1] + X[n-2] + \ldots + X[n-7]$$

   b. (5) Find the mean of the random process $Y[n]$.

   c. (7) Find the cross-correlation $r_{XY}[n]$ between $X[n]$ and $Y[n]$.

   d. (8) Find the autocorrelation $r_{YY}[n]$ for the process $Y[n]$.

a) $$r_{xx}[n] = E[X(l)X(l+n)] = \begin{cases} E[X(l)]E[X(l+n)], & n \neq 0 \\ E[X(l)^2], & n = 0 \end{cases}$$

$$= \begin{cases} 1 \cdot 1, & n \neq 0 \\ \sigma_X^2 + E[X(l)]^2, & n = 0 \end{cases}$$

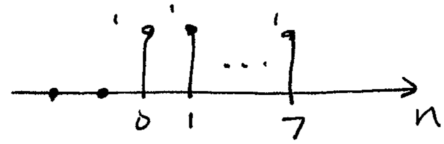
$$= \begin{cases} 1, & n \neq 0 \\ 2, & n = 0 \end{cases}$$

$$\boxed{r_{xx}[n] = 1 + \delta[n]}$$

b) $$E[Y(n)] = E\left[\sum_{k=0}^{7} X(n-k)\right] = \sum_{k=0}^{7} E[X(n-k)] = \sum_{k=0}^{7} 1 = \boxed{8}$$

$$\boxed{E[Y[n]] = 8}$$ for all n

c) $h[n] = \delta[n] + \delta[n-1] + \cdots + \delta[n-7]$

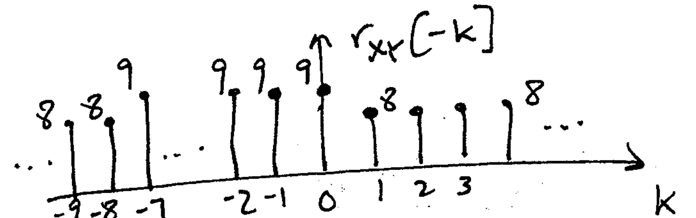


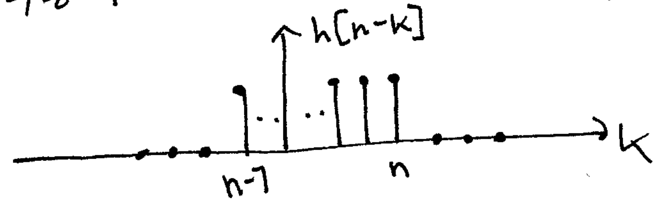
$$r_{XY}[n] = h[n] * r_{XX}[n] = h[n] * \{1 + \delta[n]\}$$
$$= h[n]*1 + h[n]*\delta[n]$$
$$= 8 + h[n]$$

$$\boxed{r_{XY}[n] = 8 + \delta[n] + \delta[n-1] + \cdots + \delta[n-7]}$$

d) $r_{YY}[n] = h[n] * r_{XY}[-n]$

$$= \sum_{k=-\infty}^{\infty} r_{XY}[-k]\, h[n-k]$$



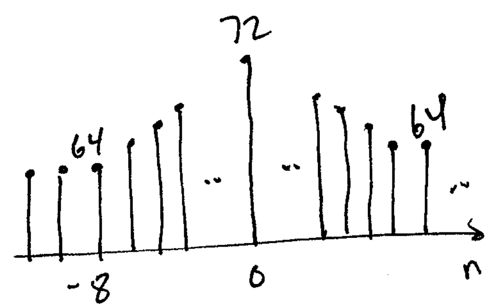


for $\begin{pmatrix} n \le -8 \\ \text{or } n \ge 8 \end{pmatrix}$, $r_{YY}[n] = 8 + 8 + \cdots + 8 = 64$

for $-8 < n \le 0$, $r_{YY}[n] = 8(8) + 1(n-(-7)+1) = 64 + (n+8)$

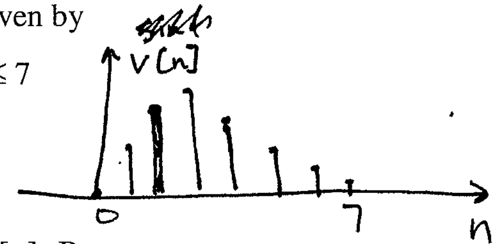
for $0 < n < 8$, $r_{YY}[n] = 8(8) + 1(0-(n-7)+1) = 64 + (8-n)$

$$\boxed{r_{YY}[n] = \begin{cases} 64, & n \le -8 \\ 64 + (n+8), & -8 < n \le 0 \\ 64 + (8-n), & 0 < n < 8 \\ 64, & n \ge 8 \end{cases}}$$



2. (35 pts.) Consider a model for a voiced phoneme of sampled speech in which the pitch period is 16 samples; and the vocal tract response is given by

$$v[n] = \begin{cases} \sin(\pi n/8) + 0.5\sin(\pi n/4), & 0 \le n \le 7 \\ 0, & \text{else} \end{cases}$$



Hint: See the formula sheet for the details of this model.

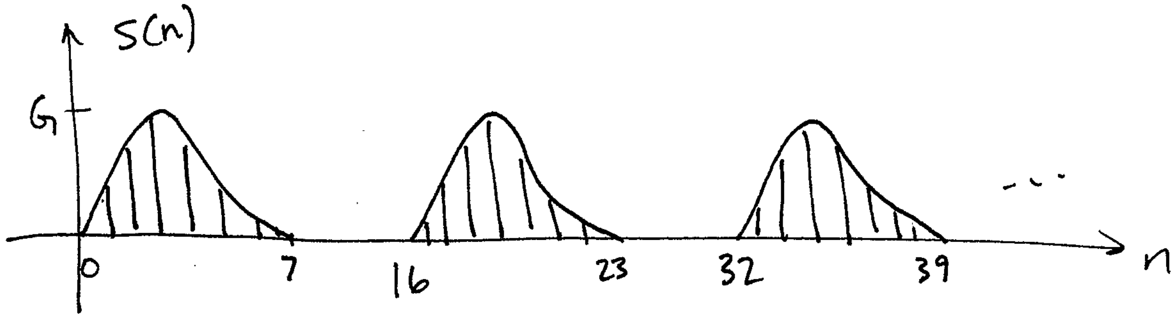
a. (10) Carefully sketch the resulting speech waveform $s[n]$. Be sure to dimension important quantities.

b. (15) Find an expression for the DTFT $S(\omega)$ of the speech waveform $s[n]$, and carefully sketch it. Be sure to label the pitch frequency and formant frequencies.

c. (5) Suppose we compute the spectrogram for this waveform using a window of length $N = 128$ samples. Sketch the resulting spectrogram. Be sure to dimension both axes.

d. (5) Repeat part c) for a window of length $N = 8$ samples.

a)



b) Assume gain $G = 1$.

$$s[n] = \sum_{k=-\infty}^{\infty} v[n-16k]$$

$$S(\omega) = \text{DTFT}\left\{\sum_{k=-\infty}^{\infty} v[n-16k]\right\}$$

$$= \sum_k \text{DTFT}\{v[n-16k]\}$$

$$= \sum_k V(\omega) e^{-j\omega 16 k}$$

$$= V(\omega) \sum_{k=-\infty}^{\infty} e^{-j(16\omega)k}$$

$$= V(\omega)\left\{\frac{1}{16}\sum_{\ell=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi \ell}{16}\right)\right\}$$

note: $1 \overset{\text{DTFT}}{\Longleftrightarrow} \sum_k e^{-j\omega k} = \text{rep}_{2\pi}\{\delta(\omega)\}$

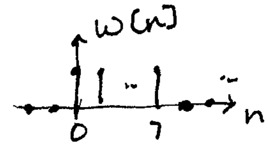
$$= \sum_{\ell=-\infty}^{\infty} \delta(\omega - 2\pi \ell)$$

So $\sum_k e^{-j(16\omega)k} = \sum_{\ell=-\infty}^{\infty} \delta(16\omega - 2\pi\ell)$

$$= \sum_\ell \delta\left(16\left(\omega - \frac{2\pi\ell}{16}\right)\right)$$

$$= \frac{1}{16}\sum_\ell \delta\left(\omega - \frac{2\pi\ell}{16}\right)$$

2. (continued) $v[n] = \{\sin(\pi n/8) + \frac{1}{2}\sin(\pi n/4)\}\cdot w[n]$



b) continued

$$v[n] = \left[\frac{1}{2j}\left\{e^{j\pi/8\, n} - e^{-j\pi/8\, n}\right\} + \frac{1}{4j}\left\{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right\}\right]\cdot w[n]$$

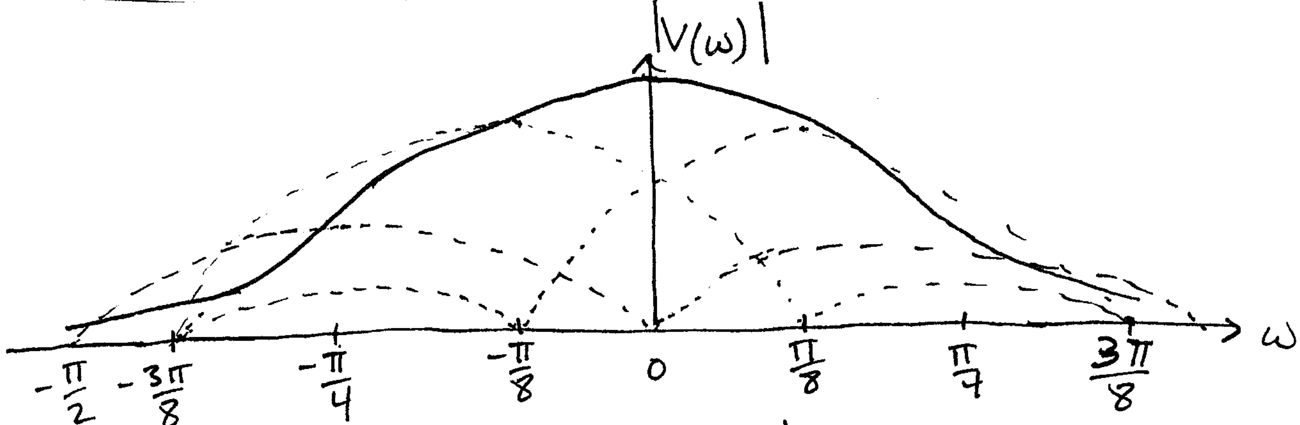
$\frac{1}{j} = -j = e^{-j\frac{\pi}{2}}$

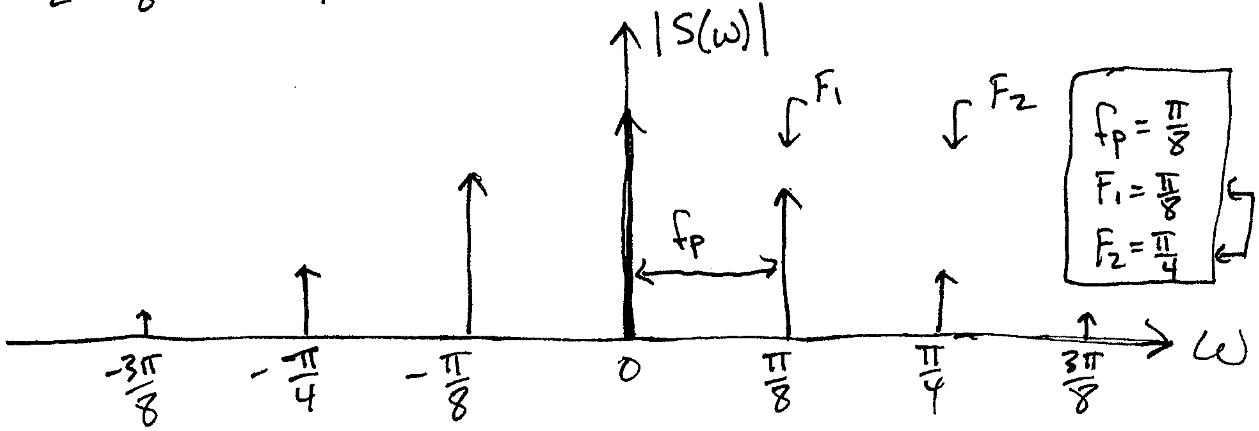
$$V(\omega) = \left[\frac{2\pi e^{-j\pi/2}}{2}\left\{\delta(\omega-\tfrac{\pi}{8}) - \delta(\omega+\tfrac{\pi}{8})\right\} + \frac{2\pi e^{-j\frac{\pi}{2}}}{4}\left\{\delta(\omega-\tfrac{\pi}{4}) - \delta(\omega+\tfrac{\pi}{4})\right\}\right] \circledast \frac{\sin(\omega 8/2)}{\sin(\omega/2)} e^{-j(\frac{7}{2})\omega}$$

$$V(\omega) = \boxed{e^{-j\pi/2}\left\{\frac{1}{2}\frac{\sin(4(\omega-\frac{\pi}{8}))}{\sin((\omega-\frac{\pi}{8})/2)} e^{-j\frac{7}{2}(\omega-\frac{\pi}{8})} - \frac{1}{2}\frac{\sin(4(\omega+\frac{\pi}{8}))}{\sin((\omega+\frac{\pi}{8})/2)} e^{-j\frac{7}{2}(\omega+\frac{\pi}{8})} + \frac{1}{4}\frac{\sin(4(\omega-\frac{\pi}{4}))}{\sin((\omega-\frac{\pi}{4})/2)} e^{-j\frac{7}{2}(\omega-\frac{\pi}{4})} - \frac{1}{4}\frac{\sin(4(\omega+\frac{\pi}{4}))}{\sin((\omega+\frac{\pi}{4})/2)} e^{-j\frac{7}{2}(\omega+\frac{\pi}{4})}\right\}}$$

$$\boxed{S(\omega) = \frac{1}{16}\,\text{comb}_{\frac{2\pi}{16}}\{V(\omega)\}}$$

↑ spacing between nulls is $\frac{\pi}{4}$.





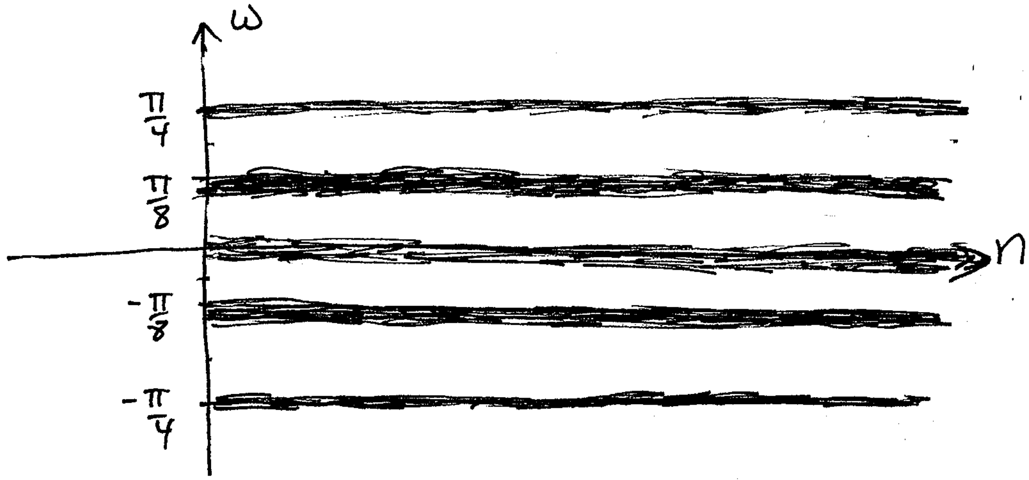
$f_p = \frac{\pi}{8}$

$F_1 = \frac{\pi}{8}$

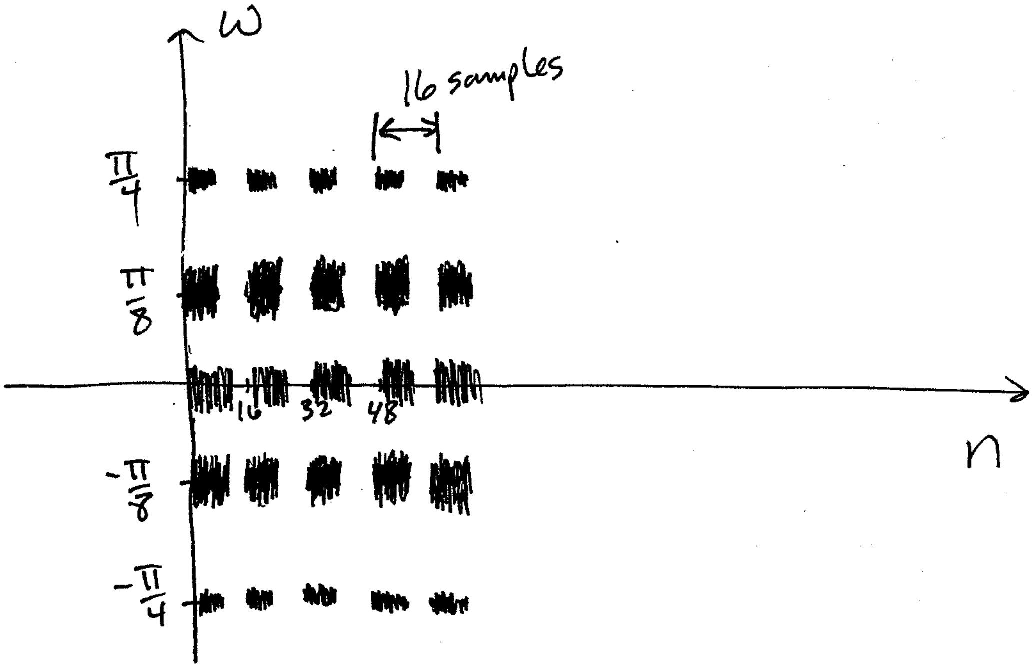
$F_2 = \frac{\pi}{4}$

sinusoid frequencies

c) Spectrogram, $N=128$



d) Spectrogram, $N=8$



3. (15) Consider a filter with frequency response

$$H_0(\omega) = 1 - |\omega|/\pi, \ |\omega| < \pi,$$

which repeats periodically outside this range.

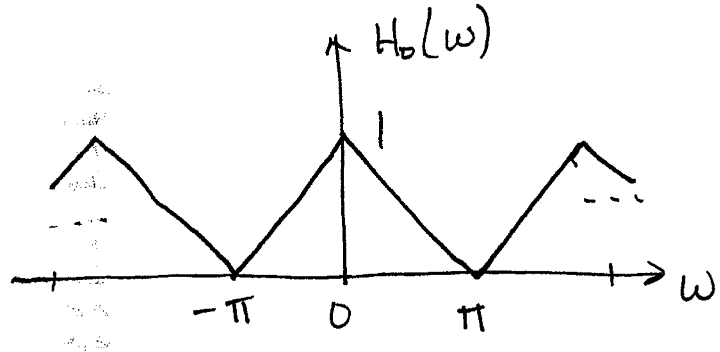
a. (5) Sketch $H_0(\omega)$

Suppose that we use $H_0(\omega)$ as the base-band filter in a 2-channel modulated filter bank. See the formula sheet for information about modulated filter banks.

b. (5) Find an expression for the frequency response $H_1(\omega)$ of the filter used for the second channel of the filter bank.

c. (5) Show *in the frequency domain* that this system ($H_0(\omega)$ and $H_1(\omega)$) will yield perfect reconstruction.

a)



b) $H_1(\omega) = H_0\left(\omega - \frac{2\pi(1)}{2}\right) = H_0(\omega - \pi)$

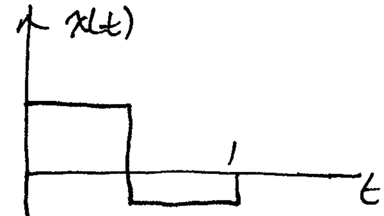
$$\boxed{H_1(\omega) = \frac{|\omega|}{\pi} \quad \text{for } |\omega| < \pi}$$

c) Need $\sum_{k=0}^{1} H_k(\omega) = 1$ for perfect reconstruction.

$H_0(\omega) + H_1(\omega) = 1 - \frac{|\omega|}{\pi} + \frac{|\omega|}{\pi} = 1$ for $|\omega| < \pi$.

So it's satisfied.

□



4. (25 pts) Consider the signal

$$x(t)=\begin{cases}0.75, & 0\le t<0.5\\ -0.25, & 0.5\le t\le 1.0\end{cases}$$

defined on the interval $0.0 \le t \le 1.0$. Suppose that we wish to approximate $x(t)$ on this interval by the function $\hat{x}(t) = at + b$, where a and b are constants.

a) (15) Find the values for a and b that will minimize the total squared error

$$\varepsilon = \int_0^1 [\hat{x}(t) - x(t)]^2\, dt$$

between $x(t)$ and $\hat{x}(t)$.

b) (5) Carefully sketch $x(t)$ and $\hat{x}(t)$ on the same set of axes for the optimal values of a and b that you determined in your answer to part a) above.

c) (5) Find the minimum value for ε that results with the optimal values of a and b that you determined in your answer to part a) above.

a) $$\varepsilon = \int_0^1 [\hat{x}(t)-x(t)]^2 dt = \int_0^1 [at+b-x(t)]^2 dt$$

$$= \int_0^{1/2} \left[at+b-\tfrac{3}{4}\right]^2 dt + \int_{1/2}^1 \left[at+b+\tfrac{1}{4}\right]^2 dt$$

$$= \int_0^{1/2} \left[a^2t^2+b^2+\tfrac{9}{16}+2abt-2b(\tfrac{3}{4})-2a(\tfrac{3}{4})t\right] dt$$

$$+ \int_{1/2}^1 \left[a^2t^2+b^2+\tfrac{1}{16}+2abt+2b(\tfrac{1}{4})+2at(\tfrac{1}{4})\right] dt$$

$$= \int_0^1 \left[a^2t^2+b^2+2abt\right] dt + \int_0^{1/2} \left[\tfrac{9}{16}-\tfrac{3}{2}b-\tfrac{3}{2}at\right] dt$$

$$+ \int_{1/2}^1 \left[\tfrac{1}{16}+\tfrac{1}{2}b+\tfrac{1}{2}at\right] dt$$

$$= \left[\tfrac{1}{3}a^2t^3+b^2t+abt^2\right]_0^1 + \left[\tfrac{9}{16}t-\tfrac{3}{2}bt-\tfrac{3}{4}at^2\right]_0^{1/2} + \left[\tfrac{1}{16}t+\tfrac{1}{2}bt+\tfrac{1}{4}at^2\right]_{\frac{1}{2}}^1$$

$$= \tfrac{1}{3}a^2+b^2+ab+\tfrac{9}{32}-\tfrac{3}{4}b-\tfrac{3}{16}a+\tfrac{1}{16}+\tfrac{1}{2}b+\tfrac{1}{4}a-\tfrac{1}{32}-\tfrac{1}{4}b-\tfrac{1}{16}a$$

$$= \tfrac{1}{3}a^2+b^2+ab-\tfrac{1}{2}b+\tfrac{10}{32}$$

4. (continued)

a) continued

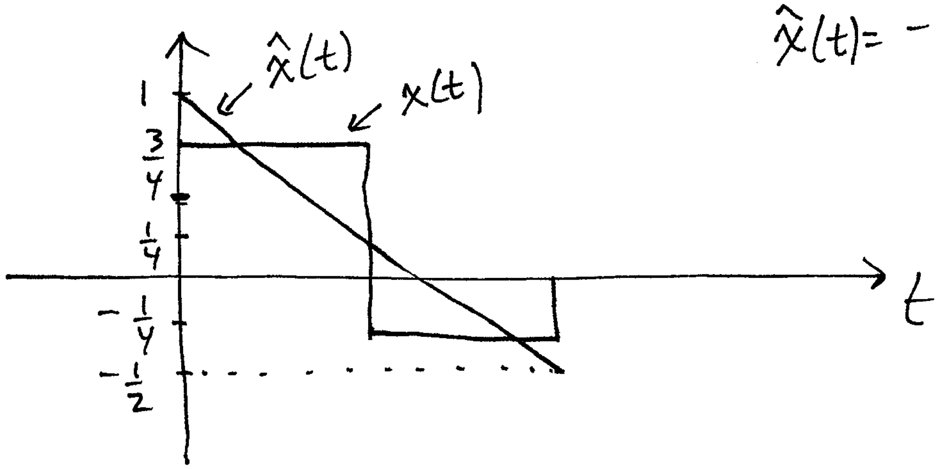
$$\frac{\partial \varepsilon}{\partial a} = \frac{2}{3}a + b \equiv 0$$

$$\frac{\partial \varepsilon}{\partial b} = 2b + a - \frac{1}{2} \equiv 0$$

$$2b + \frac{4}{3}a = 0$$

$$\Rightarrow \boxed{a = -\frac{3}{2} \quad b = 1}$$

b)



$$\hat{x}(t) = -\frac{3}{2}t + 1$$

c)
$$\varepsilon_{min} = \frac{1}{3}\left(-\frac{3}{2}\right)^2 + (1)^2 + \left(-\frac{3}{2}\right)(1) - \frac{1}{2}(1) + \frac{10}{32}$$

$$= \frac{1}{\cancel{3}} \cdot \frac{\cancel{9}^{3}}{4} + 1 - \frac{3}{2} - \frac{1}{2} + \frac{10}{32}$$

$$= \boxed{\frac{1}{16}}$$

Alternate solutions for (a) and (c) (differentiate without getting explicit expression for error)

a) $$\mathcal{E} = \int_0^{1/2} \left[at + b - \tfrac{3}{4}\right]^2 dt + \int_{1/2}^{1} \left[at + b + \tfrac{1}{4}\right]^2 dt$$

$$\frac{\partial}{\partial a}\mathcal{E} = \int_0^{1/2} 2\left[at + b - \tfrac{3}{4}\right] t \, dt + \int_{1/2}^{1} 2\left[at + b + \tfrac{1}{4}\right] t \, dt$$

$$= \int_0^{1/2} \left[2at^2 + 2bt - \tfrac{3}{2}t\right] dt + \int_{1/2}^{1} \left[2at^2 + 2bt + \tfrac{1}{2}t\right] dt$$

$$= \left[\tfrac{2}{3}at^3 + bt^2 - \tfrac{3}{4}t^2\right]_0^{1/2} + \left[\tfrac{2}{3}at^3 + bt^2 + \tfrac{1}{4}t^2\right]_{1/2}^{1}$$

$$= \cancel{\tfrac{2}{3\cdot 8}a} + \cancel{\tfrac{1}{4}b} - \tfrac{3}{16} + \tfrac{2}{3}a + b + \tfrac{1}{4} - \cancel{\tfrac{2}{3\cdot 8}a} - \cancel{\tfrac{1}{4}b} - \tfrac{1}{16}$$

$$= \tfrac{2}{3}a + b \equiv 0 \qquad (*)$$

$$\frac{\partial}{\partial b}\mathcal{E} = \int_0^{1/2} 2\left[at + b - \tfrac{3}{4}\right] dt + \int_{1/2}^{1} 2\left[at + b + \tfrac{1}{4}\right] dt$$

$$= at^2\Big|_0^1 + 2\left[bt - \tfrac{3}{4}t\right]_0^{1/2} + \left[2bt + \tfrac{1}{2}t\right]_{1/2}^{1}$$

$$= a + b - \tfrac{3}{4} + 2b + \tfrac{1}{2} - b - \tfrac{1}{4}$$

$$= a + 2b - \tfrac{1}{2} \equiv 0 \qquad (**)$$

Solving $(*)$ and $(**)$ yields $$\boxed{\begin{aligned} a &= -\tfrac{3}{2} \\ b &= 1 \end{aligned}}$$

Alternate Solution (compute error integral after optimal coeffs. known)

c) $$\mathcal{E}_{min} = \int_0^{1/2} \left[-\frac{3}{2}t + \frac{1}{4}\right]^2 dt + \int_{1/2}^{1} \left[-\frac{3}{2}t + \frac{5}{4}\right]^2 dt$$

$$= \int_0^{1/2} \left[\frac{9}{4}t^2 - \frac{3}{4}t + \frac{1}{16}\right] dt + \int_{1/2}^{1} \left[\frac{9}{4}t^2 - \frac{15}{4}t + \frac{25}{16}\right] dt$$

$$= \frac{9}{12}t^3\Big|_0^1 + \left[-\frac{3}{8}t^2 + \frac{1}{16}t\right]_0^{1/2} + \left[-\frac{15}{8}t^2 + \frac{25}{16}t\right]_{1/2}^{1}$$

$$= \frac{9}{12} - \frac{3}{32} + \frac{1}{32} - \frac{15}{8} + \frac{25}{16} + \frac{15}{32} - \frac{25}{32}$$

$$= \frac{24 - 3 + 1 - 60 + 50 + 15 - 25}{32}$$

$$= \boxed{\frac{1}{16}}$$