21 March 2007 **Name:** Solution

ECE 438 **Exam No. 2** **Spring 2007**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider a discrete-time, linear, time-invariant system with transfer function

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}.$$

a. (1) Is this system stable?

b. (4) Find a difference equation that can be used to implement this system.

c. (10) Use the graphical approach to find the magnitude and phase of the frequency response at frequency $\omega = \pi/2$ radians/sample. Note that $\arctan(0.5) = 26.56°$.

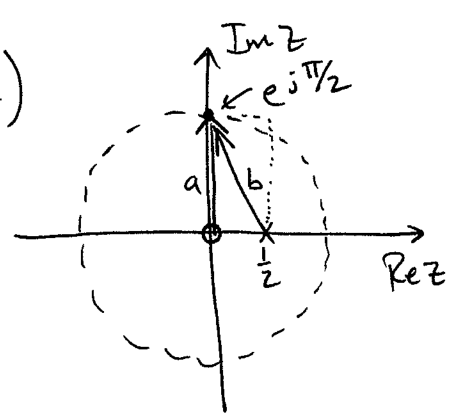
d. (10) Suppose that the output of this system is $y[n] = \left[2 - \left(\frac{1}{2}\right)^n\right]u[n]$. Find the input $x[n]$.

a) Yes. (pole at $z = \frac{1}{2}$ is inside unit circle)

b) $H(z) = \frac{Y(z)}{X(z)}$ $\quad Y(z)\left(1 - \frac{1}{2}z^{-1}\right) = X(z)$

$$\boxed{y[n] = x[n] + \frac{1}{2}y[n-1]}$$

c)



$$|H(\tfrac{\pi}{2})| = \frac{|a|}{|b|} = \frac{1}{\sqrt{1 + 1/4}} = \boxed{\frac{2}{\sqrt{5}}}$$

$$\angle H(\tfrac{\pi}{2}) = \angle a - \angle b = \frac{\pi}{2} - \left(\frac{\pi}{2} + \tan^{-1}\left(\tfrac{1}{2}\right)\right)$$

$$= -\tan^{-1}\left(\tfrac{1}{2}\right) = \boxed{-26.56°}$$

$H(z) = \frac{z}{z - \frac{1}{2}}$

1 d) $y[n] = \left[2 - \left(\frac{1}{2}\right)^n\right] u[n]$

$$Y(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} = X(z)H(z) = X(z)\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)$$

$$X(z) = \frac{2-z^{-1}}{1-z^{-1}} - 1 = \frac{2-z^{-1}}{1-z^{-1}} - \frac{1-z^{-1}}{1-z^{-1}}$$

$$= \frac{1}{1-z^{-1}}$$

$$x[n] = (1)^n u[n]$$

$$\boxed{x[n] = u[n]}$$

2. (25 pts.) For this problem, you should use the formulas provided in the handout as much as possible. Consider the 16-point signal $x[n]=\cos(10\pi n/16),\ n=0,\ldots,15$.

   a. (15) Find the 16-point DFT $X[k]$ of this signal.

   b. (8) Now find the DTFT $Y(\omega)$ for the signal $y[n]=\begin{cases} x[n], & n=0,\ldots,15 \\ 0, & \text{else} \end{cases}$.

   c. (2) Explain the relationship between $X[k]$ and $Y(\omega)$.

a) $x(n) = \cos(2\pi(5)n/16)$

Using transform pair

$$\boxed{X^{(16)}(k) = 8\delta(k-5) + 8\delta(k-11), \quad 0 \le k \le 15}$$

b) $y(n) = \overbrace{\cos(10\pi n/16)}^{x(n)} \cdot w(n)$ where $w[n] = \begin{cases} 1, & n=0,1,\ldots,15 \\ 0, & \text{else} \end{cases}$

$$Y(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mu)W(\omega-\mu)\,d\mu$$

$$X(\omega) = \pi\delta(\omega - \tfrac{5}{8}\pi) + \pi\delta(\omega + \tfrac{5}{8}\pi), \quad \pi \le \omega \le \pi$$

$$W(\omega) = \sum_{n=0}^{15} e^{-j\omega n} = \frac{1-e^{-j\omega 16}}{1-e^{-j\omega}} = \frac{\sin(8\omega)}{\sin(\frac{1}{2}\omega)}\frac{e^{-j\omega 8}}{e^{-j\omega/2}}$$

$$Y(\omega) = \frac{1}{2}\frac{\sin(8(\omega-\frac{5}{8}\pi))}{\sin(\frac{1}{2}(\omega-\frac{5}{8}\pi))}e^{-j(\omega-\frac{5}{8}\pi)\frac{15}{2}} + \frac{1}{2}\frac{\sin(8(\omega+\frac{5}{8}\pi))}{\sin(\frac{1}{2}(\omega+\frac{5}{8}\pi))}e^{-j(\omega+\frac{5}{8}\pi)\frac{15}{2}}$$

for $-\pi \le \omega \le \pi$.

or

$$Y(\omega) = \frac{1}{2}\frac{\sin(8(\omega-\frac{5}{8}\pi))}{\sin(\frac{1}{2}(\omega-\frac{5}{8}\pi))}e^{-j(\omega-\frac{5}{8}\pi)\frac{15}{2}} + \frac{1}{2}\frac{\sin(8(\omega-\frac{11\pi}{8}))}{\sin(\frac{1}{2}(\omega-\frac{11\pi}{8}))}e^{-j(\omega-\frac{11\pi}{8})\frac{15}{2}}$$

for $0 \le \omega \le 2\pi$

c) $X(k)$ is $Y(\omega)$ sampled at $\omega = \frac{2\pi k}{16}$, $k=0,1,\ldots,15$.

3. a. (21) Derive a decimation-in-time FFT algorithm for a 6-point DFT, and draw a *complete* flow diagram for the algorithm.

b. (4) Calculate the approximate number of complex operations required to compute the 6-point FFT and compare with the number of complex operations required to compute the 6-point DFT directly.

a) $$X^{(6)}(k) = \sum_{n=0}^{5} x[n] e^{-j2\pi kn/6} = \underbrace{\sum_{m=0}^{2} x(2m) e^{-j2\pi k(2m)/6}}^{\text{decimate by 2}} + \sum_{m=0}^{2} x(2m+1) e^{-j2\pi k(2m+1)/6}$$

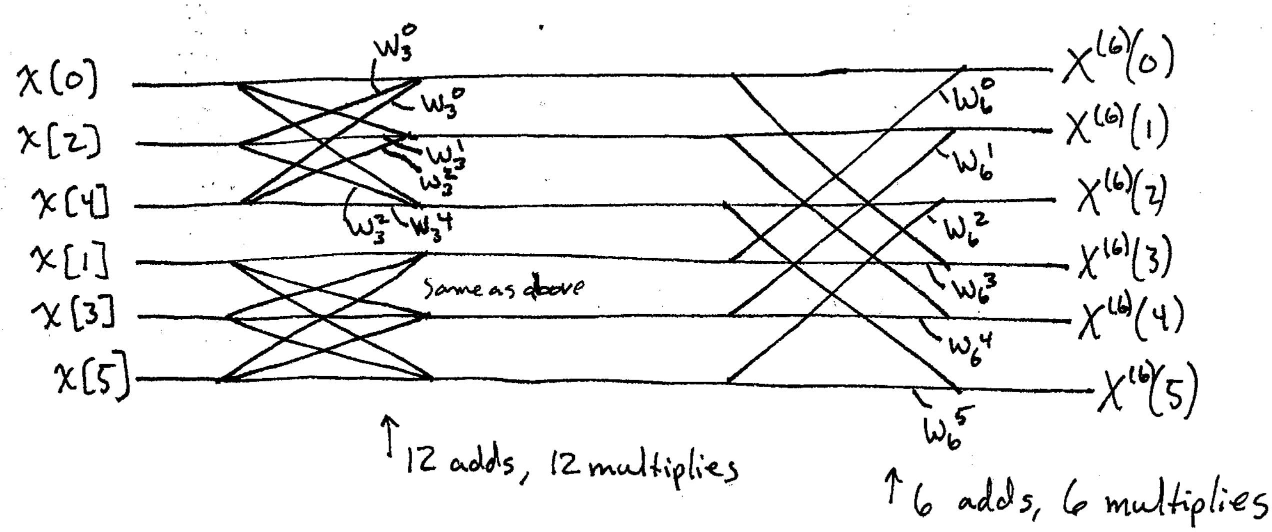
$$= X_0^{(3)}(k) + X_1^{(3)}(k) W_6^k$$

where $x_0(n) = x(2n),\ n = 0,1,2$

$x_1(n) = x(2n+1),\ n = 0,1,2$

$W_6^k = e^{-j\frac{2\pi}{6}k}$

$$X^{(3)}(k) = x(0) + x(1)e^{-j2\pi k/3} + x(2)e^{-j2\pi(2)k/3}$$



b) Consider a complex operation is 1 add and 1 multiply.

This algorithm has $\boxed{18}$ complex operations.

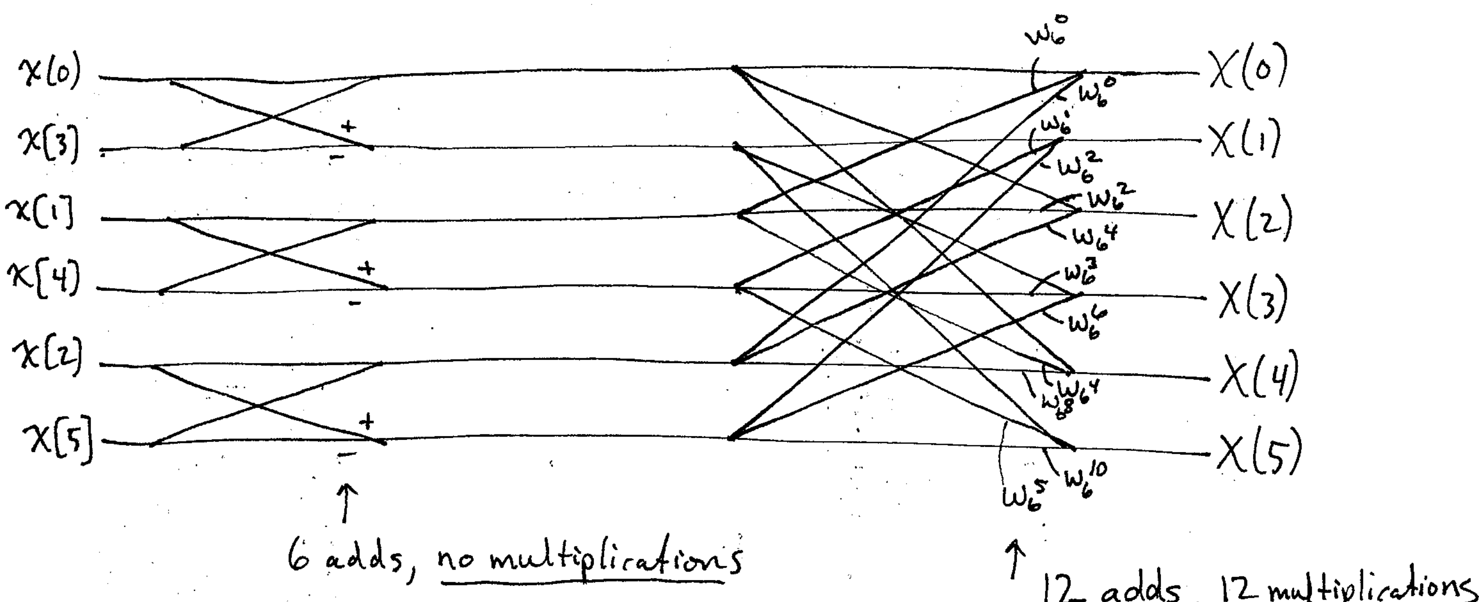
A direct DFT would have approximately $6^2 = \boxed{36}$ complex operations

3. (Alternate solution)

a) $$X^{(6)}(k) = \sum_{m=0}^{1} x(3m) e^{-j2\pi k(3m)/6} + \sum_{m=0}^{1} x(3m+1) e^{-j2\pi k(3m+1)/6} + \sum_{m=0}^{1} x(3m+2) e^{-j2\pi k(3m+2)/6}$$

$$= X_0^{(2)}(k) + X_1^{(2)}(k) W_6^k + X_2^{(2)}(k) W_6^{2k}$$

$$X^{(2)}(k) = \begin{cases} x[0]+x[1] & \text{if } k=0 \\ x[0]-x[1] & \text{if } k=1 \end{cases}$$



b) Approximately $\boxed{12}$ complex operations.

4. (25 pts) Consider a random variable X with density function

$$f_x(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

a) (8) Find the mean and variance of X.

Now suppose that we quantize X to two levels $\frac{1}{4}$ and $\frac{3}{4}$. Let $Y = Q(X)$ denote the new random variable obtained by quantizing X.

b) (8) Find the mean and variance of Y.

c) (8) Find $\mathrm{E}(XY)$.

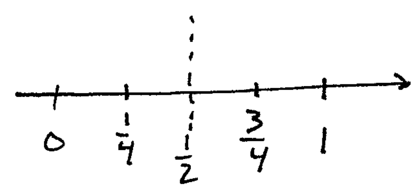
d) (1) Find the correlation coefficient ρ_{XY}. (Skip this part, if you are short on time. It is only worth one point!)

a) $$E[X] = \int_{-\infty}^{\infty} x f_x(x)\,dx = \int_0^1 2x^2\,dx = 2\frac{x^3}{3}\Big|_0^1 = \boxed{\frac{2}{3}}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_0^1 2x^3\,dx = 2\frac{x^4}{4}\Big|_0^1 = \frac{1}{2}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{1}{2} - \frac{4}{9} = \boxed{\frac{1}{18}}$$

b) $$Y = \begin{cases} \frac{1}{4}, & X \le \frac{1}{2} \\ \frac{3}{4}, & X > \frac{1}{2} \end{cases}$$



$$E[Y] = \sum_k y_k P(Y = y_k) = \frac{1}{4}P(Y=\tfrac{1}{4}) + \frac{3}{4}P(Y=\tfrac{3}{4})$$

$$= \frac{1}{4}P(X \le \tfrac{1}{2}) + \frac{3}{4}P(X > \tfrac{1}{2})$$

$$= \frac{1}{4}\int_0^{1/2} 2x\,dx + \frac{3}{4}\int_{1/2}^1 2x\,dx = \frac{1}{4}x^2\Big|_0^{1/2} + \frac{3}{4}x^2\Big|_{1/2}^1$$

$$= \frac{1}{4}\left(\frac{1}{4}\right) + \frac{3}{4} - \frac{3}{4}\left(\frac{1}{4}\right) = \boxed{\frac{5}{8}}$$

4 b) $E[Y^2] = \sum_k y_k^2 P(Y=y_k) = \left(\frac{1}{4}\right)^2 P(Y=\tfrac{1}{4}) + \left(\frac{3}{4}\right)^2 P(Y=\tfrac{3}{4})$

from prev. problem

$= \left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^2\left(\frac{3}{4}\right)$

$= \frac{1}{64} + \frac{27}{64} = \frac{28}{64}$

$\mathrm{Var}(Y) = E[Y^2] - E[Y]^2 = \frac{28}{64} - \frac{25}{64} = \boxed{\frac{3}{64}}$

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c) $E[XY] = \int_0^{1/2} \frac{1}{4}x(2x)\,dx + \int_{1/2}^{1} \frac{3}{4}x(2x)\,dx$

$XY = \begin{cases} \frac{1}{4}X, & X < \frac{1}{2} \\ \frac{3}{4}X, & X > \frac{1}{2} \end{cases}$

$= \frac{1}{2}\frac{x^3}{3}\Big|_0^{1/2} + \frac{3}{2}\frac{x^3}{3}\Big|_{1/2}^{1}$

$= \frac{1}{48} + \frac{1}{2} - \frac{1}{16} = \boxed{\frac{11}{24}}$

d) $\rho_{XY} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = \frac{\frac{11}{24} - \left(\frac{2}{3}\right)\left(\frac{5}{8}\right)}{\sqrt{1/18}\,\sqrt{3/64}} = \boxed{\sqrt{\frac{2}{3}}} = 0.8165$