14 February 2007 **Name:** Solution

ECE 438 **Exam No. 1** **Spring 2007**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the system defined by the difference equation

$$y[n] = x[n] - x[n-2].$$

a. (6) Find the response of this system to the following three inputs

n	K	-2	-1	0	1	2	3	4	5	6	7	8	9	10	K
$x_1(n)$	K	0	0	0	1	1	1	1	1	1	1	1	1	1	K
$x_2(n)$	K	0	0	1	1	0	0	1	1	0	0	1	1	0	K
$x_3(n)$	K	0	0	1	0	1	0	1	0	1	0	1	0	1	K

b. (6) Find a simple expression for the frequency response of this system.

c. (8) Find simple expressions for the magnitude and phase of the frequency response.

d. (2) Carefully sketch the magnitude of the frequency response.

e. (3) Comment on the relationship between your answers to part (a) and part (d).

a)

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$y_1[n]$	0	0	0	1	1	0	0	0	0	0	0	0	0
$y_2[n]$	0	0	1	1	-1	-1	1	1	-1	-1	1	1	-1
$y_3[n]$	0	0	1	0	0	0	0	0	0	0	0	0	0

b) $Y(\omega) = X(\omega) - X(\omega)e^{-j2\omega}$ $\quad \dfrac{Y(\omega)}{X(\omega)} = 1 - e^{-j2\omega} = e^{-j\omega}(e^{j\omega} - e^{-j\omega})$

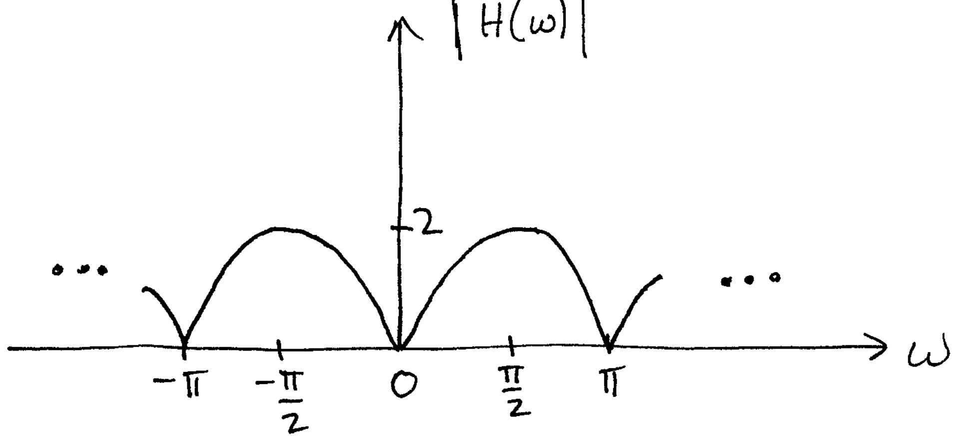
$$= 2j\sin(\omega)\, e^{-j\omega}$$

$$H(\omega) = \boxed{2\sin(\omega)\, e^{-j(\omega - \frac{\pi}{2})}}$$

1. (continued)

c) $|H(\omega)| = 2|\sin(\omega)|$ $\angle H(\omega) = \begin{cases} -\omega + \frac{\pi}{2} &, \sin(\omega) \geq 0 \\ -\omega + \frac{\pi}{2} + \pi &, \sin(\omega) < 0 \end{cases}$

d)

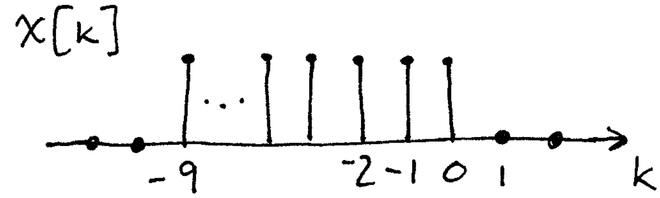


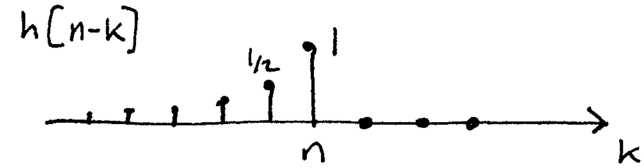
e)
- $x_2[n]$ is periodic (in the shown interval) with period 4, so it will have a strong frequency component $\cos(2\pi(\frac{1}{4})n)$. $= \cos(\frac{\pi}{2}n)$. This component at $\omega = \frac{\pi}{2}$ is amplified by 2, which is reflected in $y_2[n]$ and in $|H(\frac{\pi}{2})|$.
- $x_3[n] = \frac{1}{2}(\cos(\pi n) + 1)$ for $n \geq 0$, and since $|H(\pi)| = 0$, the oscillation is filtered out.
- $x_1[n]$ is a dc value for $n \geq 1$, and since $|H(0)| = 0$, the dc component is removed.

2. (25 pts.) Consider a linear, time-invariant system with input
$$x[n] = \begin{cases} 1, & -9 \le n \le 0 \\ 0, & \text{else} \end{cases}$$ and impulse response $h[n] = 2^{-n}u[n]$.

a. (20) Use convolution to find output $y[n]$ from the system.

b. (5) Sketch the output $y[n]$.





a) $$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$\underline{n < -9}$: $y[n] = \underline{0}$

$\underline{-9 \le n \le 0}$: $$y[n] = \sum_{k=-9}^{n} 2^{-(n-k)} = 2^{-n}\sum_{k=-9}^{n} 2^{k} = 2^{-n}\frac{2^{-9} - 2^{n+1}}{1-2}$$

$$= 2^{-n}\left(2^{n+1} - 2^{-9}\right) = \underline{2 - 2^{-(n+9)}}$$

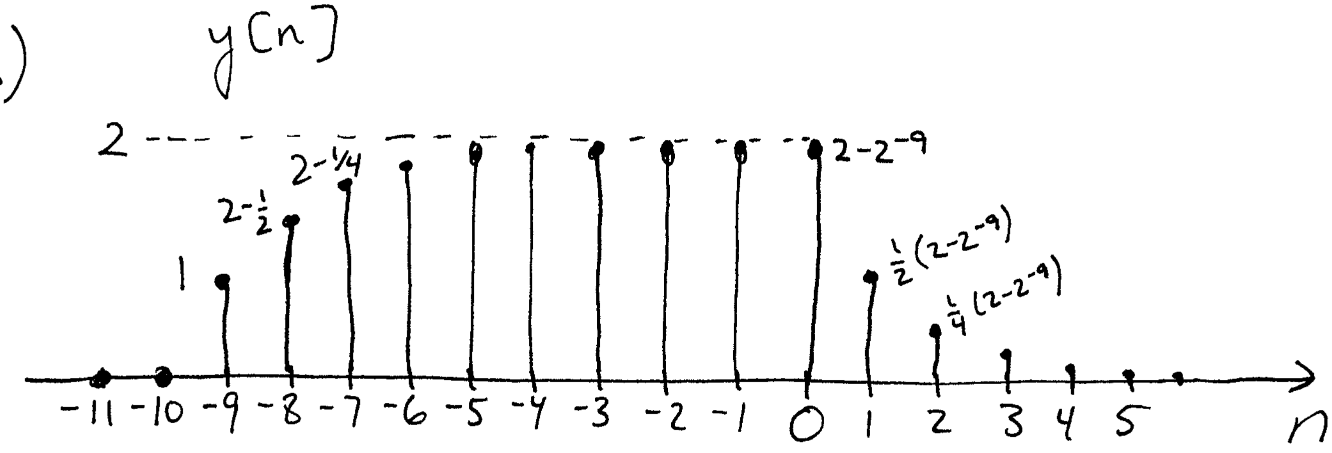
$\underline{n > 0}$: $$y[n] = \sum_{k=-9}^{0} 2^{-(n-k)} = 2^{-n}\sum_{k=-9}^{0} 2^{k} = 2^{-n}\frac{2^{-9} - 2^{1}}{1-2}$$

$$= 2^{-n}\left(2 - 2^{-9}\right)$$

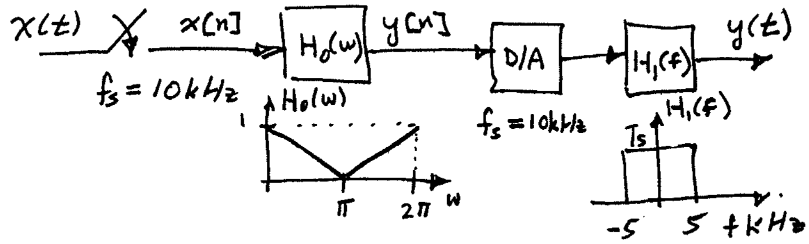
$$y[n] = \begin{cases} 0, & n < -9 \\ 2 - 2^{-(n+9)}, & -9 \le n \le 0 \\ 2^{-n}\left(2 - 2^{-9}\right), & n > 0 \end{cases}$$

2. (continued)

b)



3. (25) Consider the system shown below.



The sampler at the input is an ideal sampler. The combination of the D/A converter and the filter $H_1(f)$ at the output can also be considered to be ideal.

Suppose the input is $x(t) = 2\cos(2\pi(1000t) + \cos(2\pi(6000t)$

a. (7) Find an expression for the DTFT $X(\omega)$ of $x[n]$.

b. (3) Sketch $X(\omega)$.

c. (7) Find an expression for the DTFT $Y(\omega)$ of $y[n]$.

d. (8) Find an expression for the output $y(t)$ of the system.

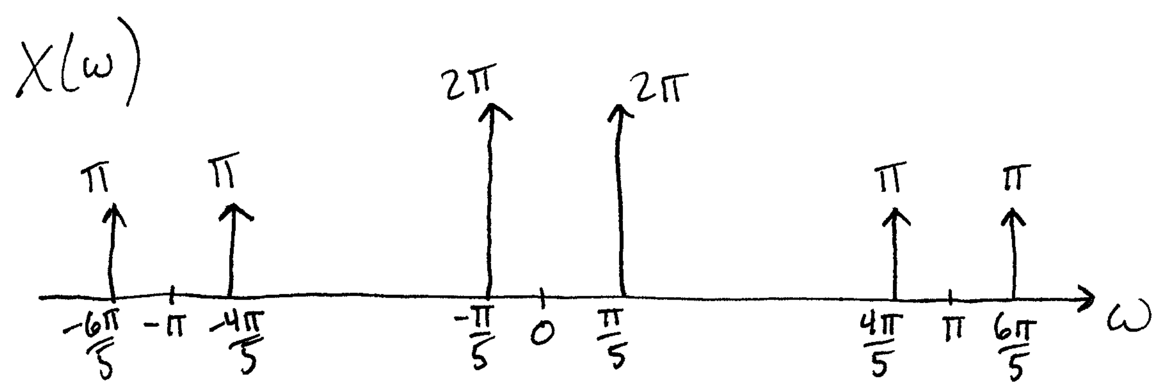
a) $$x[n] = x(t)\big|_{t=nT_s = n/f_s} = 2\cos\left(2\pi\left(\frac{1000}{10,000}\right)n\right) + \cos\left(2\pi\left(\frac{6000}{10,000}\right)n\right)$$

$$= 2\cos\left(\frac{\pi}{5}n\right) + \cos\left(\frac{6\pi}{5}n\right)$$

$$X(\omega) = \pi\, \text{rep}_{2\pi}\left\{2\delta\left(\omega-\frac{\pi}{5}\right) + 2\delta\left(\omega+\frac{\pi}{5}\right) + \delta\left(\omega-\frac{6\pi}{5}\right) + \delta\left(\omega+\frac{6\pi}{5}\right)\right\}$$

or $$= 2\pi\delta\left(\omega-\frac{\pi}{5}\right) + 2\pi\delta\left(\omega+\frac{\pi}{5}\right) + \pi\delta\left(\omega-\frac{4\pi}{5}\right) + \pi\delta\left(\omega+\frac{4\pi}{5}\right)$$

for $-\pi < \omega < \pi$

b) 

3. (continued)

c) $$Y(\omega) = X(\omega) H_0(\omega)$$

$$H_0(\omega) = 1 - \frac{|\omega|}{\pi}, \; -\pi \le \omega \le \pi$$

$$H_0\left(\frac{\pi}{5}\right) = 4/5, \quad H_0\left(\frac{4\pi}{5}\right) = \frac{1}{5}$$

$$\boxed{Y(\omega) = \frac{8\pi}{5}\delta\left(\omega - \frac{\pi}{5}\right) + \frac{8\pi}{5}\delta\left(\omega + \frac{\pi}{5}\right) + \frac{\pi}{5}\delta\left(\omega - \frac{4\pi}{5}\right) + \frac{\pi}{5}\delta\left(\omega + \frac{4\pi}{5}\right) \quad \text{for } -\pi \le \omega \le \pi}$$

d) $$Y(f) = H_1(f) \cdot Y(\omega)\Big|_{\omega = \frac{2\pi f}{f_s}}$$

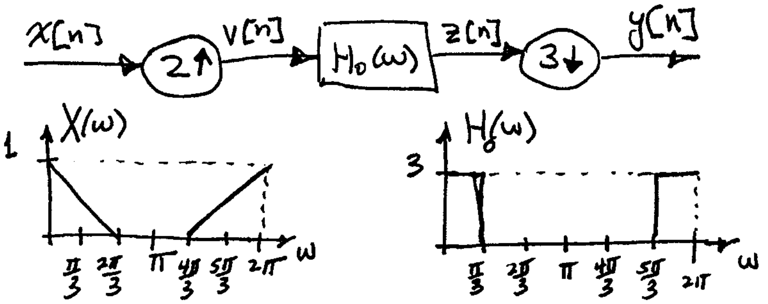
note: $$\delta(\omega - \omega_0)\Big|_{\omega = 2\pi f/f_s} = \delta\left(\frac{2\pi f}{f_s} - \frac{2\pi}{f_s}\left(\frac{\omega_0}{2\pi/f_s}\right)\right) = \delta\left(\frac{2\pi}{f_s}\left(f - \frac{\omega_0}{2\pi/f_s}\right)\right) = \frac{1}{2\pi T_s}\delta\left(f - \frac{\omega_0}{2\pi} f_s\right)$$

$$= T_s\left\{\frac{8\pi}{5}\left(\frac{1}{2\pi T_s}\right)\left[\delta(f-1000) + \delta(f+1000)\right] + \frac{\pi}{5}\left(\frac{1}{2\pi T_s}\right)\left[\delta(f-4000) + \delta(f+4000)\right]\right\}$$

$$= \frac{4}{5}\delta(f-1000) + \frac{4}{5}\delta(f+1000) + \frac{1}{10}\delta(f-4000) + \frac{1}{10}\delta(f+4000)$$

So $$\boxed{y(t) = \frac{8}{5}\cos(2\pi(1000t)) + \frac{1}{5}\cos(2\pi(4000)t)}$$

4. (25 pts) Consider the DT system shown below where the input $x[n]$ has the DTFT $X(\omega)$ shown below and the filter has the frequency response $H_0(\omega)$ also shown below.



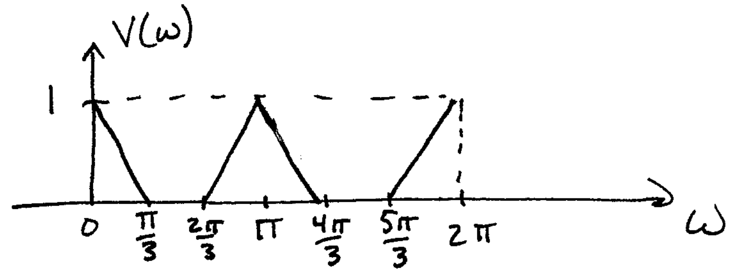
a) (8) Find an expression for the DTFT $V(\omega)$ of the signal $v[n]$ in terms of $X(\omega)$. Sketch $V(\omega)$. Be sure to make your sketch accurate and complete.

b) (8) Find an expression for the DTFT $Z(\omega)$ of the signal $z[n]$ in terms of $X(\omega)$. Sketch $Z(\omega)$. Be sure to make your sketch accurate and complete.

c) (8) Find an expression for the DTFT $Y(\omega)$ of the signal $y[n]$ in terms of $X(\omega)$. Sketch $Y(\omega)$. Be sure to make your sketch accurate and complete.

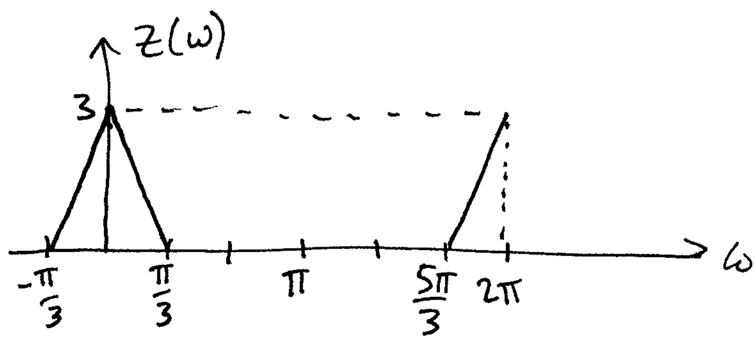
d) (1) Describe in simple terms what is the overall function of this system.

a) $V(\omega) = X(2\omega)$



b) $Z(\omega) = 3V(\omega)\,\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)$ for $-\pi < \omega < \pi$

$= \boxed{3X(2\omega)\,\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right) \text{ for } -\pi < \omega < \pi}$



4. (continued)

c) $$Y(\omega) = \frac{1}{3}\sum_{k=0}^{2} Z\left(\frac{\omega-2\pi k}{3}\right) = \frac{1}{3}Z\left(\frac{\omega}{3}\right) + \frac{1}{3}Z\left(\frac{\omega-2\pi}{3}\right) + \frac{1}{3}Z\left(\frac{\omega-4\pi}{3}\right)$$

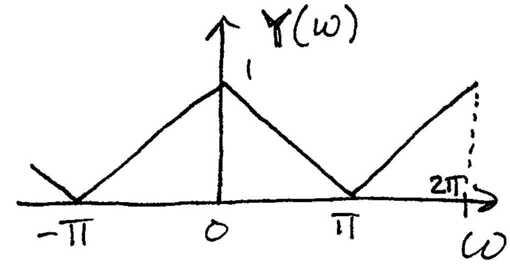
$$= X\left(2\left(\frac{\omega}{3}\right)\right)\mathrm{rect}\left(\frac{\omega/3}{2\pi/3}\right) \rightarrow = 1 \text{ in } -\pi < \omega < \pi.$$

$$+ X\left(2\left(\frac{\omega-2\pi}{3}\right)\right)\mathrm{rect}\left(\frac{\frac{\omega-2\pi}{3}}{2\pi/3}\right)$$

$$+ X\left(2\left(\frac{\omega-4\pi}{3}\right)\right)\mathrm{rect}\left(\frac{\frac{\omega-4\pi}{3}}{2\pi/3}\right)$$

These terms $= 0$ in $-\pi < \omega < \pi$.

$$\boxed{Y(\omega) = X\left(\tfrac{2}{3}\omega\right) \quad \text{for } -\pi < \omega < \pi}$$



d) The system decreases the effective sampling rate by $\frac{2}{3}$.

Alternatively, the system stretches the spectra in the range $\omega \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ to $[-\pi, \pi]$.