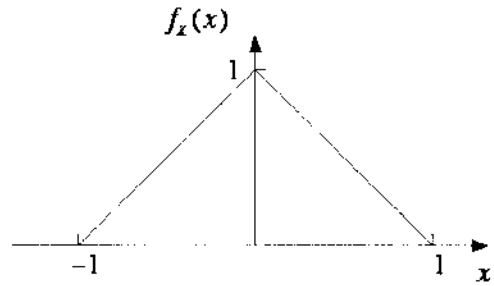
19 April 2004 **Name:**____________________

ECE 438 **Exam No. 3** **Spring 2004**

- You have 60 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider a random variable X with probability density function $f_X(x)$ shown below.



a. (8) Find the mean μ_X and variance σ_X^2 for this random variable.
b. (6) Suppose that we quantize X to the two output levels −1/3 and 1/3. Find the mean square quantization error using the approximate expression derived in class.
c. (2) Find the signal-to-noise ratio due quantization based on your answers above.
d. (9) Find the exact mean-square quantization error.

a) $\boxed{\mu_X = 0}$

$$\sigma_X^2 = E[X^2] = \int_{-1}^{0} x^2(1+x)\,dx + \int_0^1 x^2(1-x)\,dx$$

$$= \int_{-1}^{0} (x^2+x^3)\,dx + \int_0^1 (x^2-x^3)\,dx$$

$$= \left(\frac{x^3}{3}+\frac{x^4}{4}\right|_{-1}^{0} + \left(\frac{x^3}{3}-\frac{x^4}{4}\right|_0^1 = \frac{1}{3}-\frac{1}{4}+\frac{1}{3}-\frac{1}{4} = \frac{1}{6}$$

$$\boxed{\sigma_X^2 = \frac{1}{6}}$$

b) $\Delta = 1$ $\quad \sigma_e^2 = \frac{\Delta^2}{12} = \boxed{\frac{1}{12}}$

c) $SNR = 10\log_{10}\left(\frac{\sigma_X^2}{\sigma_e^2}\right) = 10\log_{10}(2) = \boxed{3.01 \text{ dB}}$

1. (continued)

d) $\epsilon_{MS} = \int (Q(x) - x)^2 f_X(x)\,dx$

$$= \int_{-1}^{0} \left(-\tfrac{1}{2} - x\right)^2 (1+x)\,dx + \int_{0}^{1} \left(\tfrac{1}{2} - x\right)^2 (1-x)\,dx$$

$$= 2 \int_{-1}^{0} \left(-\tfrac{1}{2} - x\right)^2 (1+x)\,dx$$

due to symmetry

$$= 2 \int_{-1}^{0} \left(\tfrac{1}{4} + x + x^2\right)(1+x)\,dx$$

$$= 2 \int_{-1}^{0} \left(\tfrac{1}{4} + \tfrac{1}{4}x + x + x^2 + x^2 + x^3\right)dx$$

$$= 2 \int_{-1}^{0} \left(\tfrac{1}{4} + \tfrac{5}{4}x + 2x^2 + x^3\right)dx$$

$$= 2 \left(\tfrac{1}{4}x + \tfrac{5}{4}\cdot\tfrac{x^2}{2} + \tfrac{2x^3}{3} + \tfrac{x^4}{4}\right)\Big|_{-1}^{0}$$

$$= 2\left(\tfrac{1}{4} - \tfrac{5}{8} + \tfrac{2}{3} - \tfrac{1}{4}\right) = 2\left(\tfrac{-15+16}{24}\right) = 2\left(\tfrac{1}{24}\right) = \tfrac{1}{12}$$

$$\Rightarrow \boxed{\epsilon_{MS} = \tfrac{1}{12}}$$

2. (25 pts.)

Consider two random variables X and Y with mean zero, variance unity, and correlation coefficient ρ_{XY}. Let us define two new random variables U and V as follows:

$$U = X + Y$$
$$V = X - Y$$

(a) (8) Find the means and variances of U and V in terms of ρ_{XY}.

(b) (6) Find the correlation coefficient ρ_{UV} for U and V in terms of ρ_{XY}.

(c) (2) Show that it is possible to recover X and Y from U and V.

(d) (5) Suppose that we wish to quantize X and Y each at a fixed level of quantization noise power 10^{-4} Further, suppose we set the range of each quantizer to be 6 times the standard deviation of the random variable being quantized, i.e from -3σ to 3σ, where σ is the standard deivation of the random variable being quantized. How many bits will be required to quantize both X and Y?

(e) (3) Now assume that $|\rho_{XY}| = 0.9$, and suppose that instead, we quantize U and V, in the same manner as we quantized X and Y in part (d) above, i.e. we want the quantization noise power to be 10^{-4}; and we set the range of each quantizer to be 6 times the standard deviation of the random variable being quantized. How many bits will be required to quantize both U and V?

(f) (1) Discuss the significance of your results.

a) $\mu_u = E[X+Y] = 0$

$\mu_v = E[X-Y] = 0$

$\sigma_u^2 = E[u^2] = E[X^2+2XY+Y^2] = 2 + 2E[XY]$

$\sigma_v^2 = E[V^2] = E[X^2-2XY+Y^2] = 2 - 2E[XY]$

$$\rho_{XY} = \frac{E[XY]-\mu_X\mu_Y}{\sigma_X\sigma_Y} = E[XY]$$

$$\Rightarrow \boxed{\begin{aligned}&\mu_u = \mu_v = 0\\ &\sigma_u^2 = 2(1+\rho_{XY})\\ &\sigma_v^2 = 2(1-\rho_{XY})\end{aligned}}$$

2. (continued)

b) $\rho_{uv} = \dfrac{E[UV] - \mu_u \mu_v}{\sigma_u \sigma_v} = \dfrac{E[UV]}{\sigma_u \sigma_v}$

$E[UV] = E[X^2 - Y^2] = \sigma_X^2 - \sigma_Y^2 = 0$

$\Rightarrow \boxed{\rho_{uv} = 0}$

c) $X = \dfrac{U+V}{2}$ $\qquad Y = \dfrac{U-V}{2}$

d) $\dfrac{\Delta^2}{12} = 10^{-4} \Rightarrow \Delta = \sqrt{12 \cdot 10^{-4}}$

$\Delta = \dfrac{6\sigma}{2^B} = \sqrt{12 \cdot 10^{-4}} \Rightarrow 2^B = \dfrac{6\sigma}{\sqrt{12 \cdot 10^{-4}}} = \sqrt{\dfrac{36\sigma^2}{12 \cdot 10^{-4}}} = \sqrt{3\sigma^2 \cdot 10^4}$

$B = \dfrac{\log_{10}(\sqrt{3\sigma^2 \cdot 10^4})}{\log_{10}(2)} = \dfrac{\log_{10}(3\sigma^2 \cdot 10^4)}{2\log_{10}(2)}$

Since $\sigma_x = \sigma_y = 1$, $B = 7.436$. Rounding up,

$\boxed{\text{8 bits are needed for each of X and Y, a total of 16 bits.}}$

e) If $\rho_{XY} = 0.9$, $\quad B_u = \dfrac{\log_{10}(3 \cdot 3.8 \cdot 10^4)}{2\log_{10}(2)} = 8.399 \rightarrow 9$

$B_v = \dfrac{\log_{10}(3 \cdot 0.2 \cdot 10^4)}{2\log_{10}(2)} = 6.275 \rightarrow 7$

If $\rho_{XY} = -0.9$, $B_u = 7$ and $B_v = 9$.

$\boxed{\text{A total of 16 bits is needed.}}$

f) Since the same number of total bits are needed to quantize X and Y and U and V, and X and Y are recoverable from U and V, using X and Y or U and V is essentially equivalent.

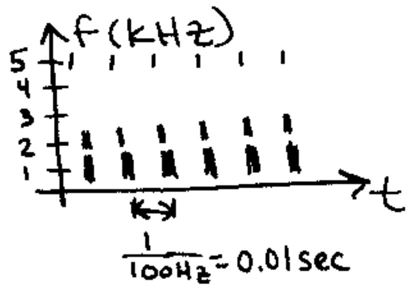
3. (25) A voiced speech waveform has pitch 100 Hz, and three formant frequencies at 1 kHz, 2 kHz, and 5 kHz, which decrease in amplitude with increasing frequency.

   a. (7) Sketch a wideband spectrogram for this waveform. Be sure to label and dimension all important quantities.

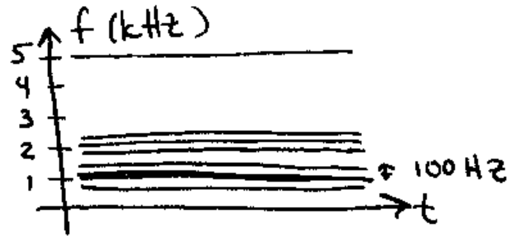
   b. (7) Sketch a narrowband spectrogram for this waveform. Be sure to label and dimension all important quantities.

   c. (11) Draw a block diagram of a digital system that could be used to synthesize this speech waveform. Assume that the system operates at a 20 kHz sampling rate. Be sure to define all components of the system and define all important parameters as accurately as possible.

a)



b)



c) $e[n] \rightarrow \boxed{V(z)} \rightarrow \boxed{\text{D/A},\ f_s = 20\text{kHz}} \rightarrow s(t)$

$$e[n] = \sum_{k=-\infty}^{\infty} \delta[n - kP]$$

$$\text{pitch period} = \frac{1}{100\text{Hz}} = \frac{P}{20\text{kHz}} \Rightarrow P = \frac{20}{.1} = 200$$

$$e[n] = \sum_{k=-\infty}^{\infty} \delta[n - k200]$$

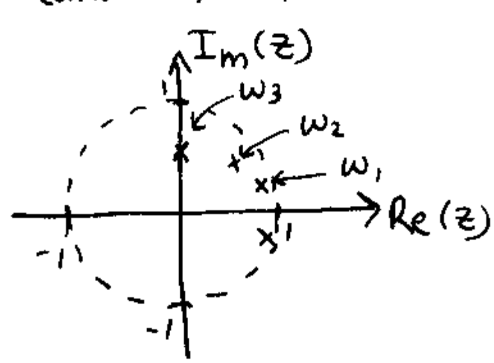
$V(z)$ has poles at frequencies ω_1, ω_2, and ω_3 (and $-\omega_1, -\omega_2, -\omega_3$).

$$\omega_1 = \frac{2\pi\, 1\text{kHz}}{20\text{kHz}} = \frac{\pi}{10}$$

$$\omega_2 = \frac{2\pi \cdot 2}{20} = \frac{\pi}{5}$$

$$\omega_3 = \frac{2\pi \cdot 5}{20} = \frac{\pi}{2}$$

distance from unit circle



4. (25 pts) Consider the signal $x(t) = (1-t)^2,\ 0 \le t \le 1$. We want to approximate this signal over the interval $0 \le t \le 1$ by the signal $\tilde{x}(t) = a_0 + a_1 t$.

a) (12) Determine values for the coefficients a_0 and a_1 that will minimize the mean-square approximation error integrated over the range $0 \le t \le 1$.

b) (7) Carefully sketch the two functions $x(t)$ and $\tilde{x}(t)$ on the same axes.

c) (6) Compute the mean-square approximation error for the coefficient values that you determined in part a) above.

a) $E = \int_0^1 (a_0 + a_1 t - (1-t)^2)^2 dt$

$= \int_0^1 (a_0 + a_1 t - (1 - 2t + t^2))^2 dt$

$= \int_0^1 [(a_0 - 1) + (a_1 + 2)t - t^2]^2 dt$

$\frac{\partial E}{\partial a_0} = 2\int_0^1 [(a_0-1) + (a_1+2)t - t^2]\, dt$

$= 2\left[(a_0-1)t + \frac{(a_1+2)t^2}{2} - \frac{t^3}{3}\right]_0^1$

$= 2\left[a_0 - 1 + \frac{a_1+2}{2} - \frac{1}{3}\right] = 2\left(a_0 + \frac{a_1}{2} - \frac{1}{3}\right) \overset{\text{set}}{=} 0$

$\Rightarrow \quad a_0 + \frac{a_1}{2} = \frac{1}{3} \qquad (*)$

$\frac{\partial E}{\partial a_1} = 2\int_0^1 [(a_0-1) + (a_1+2)t - t^2]\, t\, dt$

$= 2\int_0^1 [(a_0-1)t + (a_1+2)t^2 - t^3]\, dt$

$= 2\left[\frac{(a_0-1)t^2}{2} + \frac{(a_1+2)t^3}{3} - \frac{t^4}{4}\right]_0^1$

$= 2\left(\frac{a_0-1}{2} + \frac{a_1+2}{3} - \frac{1}{4}\right) = 2\left(\frac{a_0}{2} + \frac{a_1}{3} - \frac{1}{12}\right) \overset{\text{set}}{=} 0$

$\Rightarrow \quad \frac{a_0}{2} + \frac{a_1}{3} = \frac{1}{12} \qquad (**)$

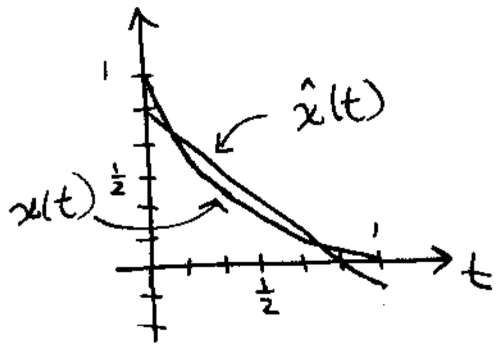
$(*) - 2(**) = -\frac{a_1}{6} = \frac{1}{6} \quad \Rightarrow \boxed{a_1 = -1}$

$a_0 = -\frac{a_1}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \quad \Rightarrow \boxed{a_0 = \frac{5}{6}}$

4. (continued)

b) $x(t) = (1-t)^2 \quad , \quad 0 \le t \le 1$

$\hat{x}(t) = \frac{5}{6} - t \quad , \quad 0 \le t \le 1$



c) $E = \int_0^1 \left[-\frac{1}{6} + t - t^2 \right]^2 dt$

$= \int_0^1 \left[\frac{1}{36} - \frac{1}{3}t + \frac{1}{3}t^2 + t^2 - 2t^3 + t^4 \right] dt$

$= \left(\frac{t}{36} - \frac{t^2}{6} + \frac{4}{3} \cdot \frac{t^3}{3} - \frac{2t^4}{4} + \frac{t^5}{5} \right|_0^1$

$= \frac{1}{36} - \frac{1}{6} + \frac{4}{9} - \frac{2}{4} + \frac{1}{5}$

$= \frac{1}{180}$

$\Rightarrow \boxed{E = \frac{1}{180} = 0.00556}$