22 March 2004 **Name:** SOLUTION

ECE 438 **Exam No. 2** **Spring 2004**

- You have 60 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the causal LTI system described by the following difference equation

$$y[n] = x[n] + \tfrac{1}{2} y[n-1]$$

Suppose that the signal $x[n] = 2^n u[-n]$ is input to this system.

a. (12) Find the Z-transform $Y(z)$ of the output $y[n]$. Be sure to state the region of convergence for $Y(z)$.

b. (2) Is the system stable? State why or why not.

c. (11) Find the system output $y[n]$ by finding the inverse Z-transform of $Y(z)$.

a) $Y(z) = X(z) + \frac{1}{2} z^{-1} Y(z)$

$$Y(z) = \frac{X(z)}{1 - \frac{1}{2} z^{-1}}$$

To find $X(z)$, use

$$-a^n u[-n-1] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}}, \quad |z|<|a|$$

$$x[n-n_0] \xleftrightarrow{ZT} z^{-n_0} X(z)$$

$$x[n] = 2^n u[-n] = 2 \cdot 2^{n-1} u[-(n-1)-1]$$

$$X(z) = \frac{-2z^{-1}}{1-2z^{-1}}$$

$$\boxed{Y(z) = \frac{-2z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{-2z}{(z-2)(z-\frac{1}{2})}}$$

$Y(z)$ has poles at $\frac{1}{2}$ and 2.

1. (continued)

Since the system is causal, we can see what the system ROC is.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} . \quad H(z) \text{ has a pole at } \frac{1}{2}.$$

The system ROC is therefore $|z| > \frac{1}{2}$ and the system is stable.

Since $x[n]$ is bounded, $y[n]$ must be bounded.

This means we must use $\boxed{\frac{1}{2} < |z| < 2}$ for the ROC of $Y(z)$.

b) The system ROC is $|z| > \frac{1}{2}$ (see part a).

Since $|z| = 1$ is contained in the system ROC, the system is stable.

c)
$$Y(z) = \frac{A}{1-2z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = Y(z)(1-2z^{-1})\Big|_{z^{-1}=\frac{1}{2}} = \frac{-1}{1-\frac{1}{4}} = -\frac{4}{3}$$

$$B = Y(z)(1-\tfrac{1}{2}z^{-1})_{z^{-1}=2} = \frac{-4}{1-4} = \frac{4}{3}$$

$$\boxed{y[n] = \frac{4}{3}(2)^n u[-n-1] + \frac{4}{3}\left(\frac{1}{2}\right)^n u[n] = \frac{4}{3}\left(\frac{1}{2}\right)^{|n|}}$$

2. (25 pts.) Consider an N-point signal $x[n], n = 0,\ldots,N-1$, where N is even. Suppose we generate a new N/2-point signal $y[n]$ by taking every other data point from $x[n]$, i.e. $y[n] = x[2n], n = 0,\ldots,\frac{N}{2}-1$.

Find a simple expression for the N/2-point DFT $Y^{(N/2)}[k]$ of $y[n]$ in terms of the N-point DFT $X^{(N)}[k]$ of $x[n]$.

$$Y^{(\frac{N}{2})}[k] = \sum_{n=0}^{\frac{N}{2}-1} x[2n] e^{-j\frac{2\pi kn}{N/2}} = \sum_{\substack{n=0 \\ n \text{ even}}}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2}(1+(-1)^n)\, x[n] e^{-j\frac{2\pi kn}{N}}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} + \frac{1}{2}\sum_{n=0}^{N-1} \underbrace{e^{j\pi n}}_{= e^{j\frac{2\pi n}{N}\left(\frac{N}{2}\right)}} x[n] e^{-j\frac{2\pi kn}{N}}$$

$$\Rightarrow \boxed{Y^{(\frac{N}{2})}[k] = \frac{1}{2} X^{(N)}[k] + \frac{1}{2} X^{(N)}[k-\tfrac{N}{2}]}$$

by the modulation property

2. (continued)

Alternate solution:

$$x[n] = \frac{1}{N}\sum_{l=0}^{N-1} X^{(N)}[l]e^{j\frac{2\pi nl}{N}} \Rightarrow x[2n] = \frac{1}{N}\sum_{l=0}^{N-1} X^{(N)}[l]e^{j\frac{2\pi\, 2nl}{N}}$$

$$Y^{(\frac{N}{2})}[k] = \sum_{n=0}^{\frac{N}{2}-1} x[2n]e^{-j\frac{2\pi kn}{N/2}}$$

$$= \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1}\sum_{l=0}^{N-1} X^{(N)}[l]e^{j\frac{2\pi nl}{N/2}}\, e^{-j\frac{2\pi kn}{N/2}}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X^{(N)}[l] \underbrace{\sum_{n=0}^{\frac{N}{2}-1} e^{j\frac{2\pi n(l-k)}{N/2}}}$$

$$\frac{1-e^{j2\pi n(l-k)}}{1-e^{j\frac{2\pi n(l-k)}{N/2}}} = \begin{cases} \frac{N}{2}, & l=\frac{N}{2}I+k, I\in\mathbb{Z} \\ 0, & \text{otherwise}\end{cases}$$

For $l\in[0,N-1]$ and $k\in[0,\frac{N}{2}-1]$, this function is $\frac{N}{2}$ for $l=k$ and $l=\frac{N}{2}+k$.

$$Y^{(\frac{N}{2})}[k] = \frac{1}{2}\sum_{l=0}^{N-1} X^{(N)}[l](\delta[l-k]+\delta[l-(\tfrac{N}{2}+k)])$$

$$\Rightarrow \boxed{Y^{(\frac{N}{2})}[k] = \frac{1}{2}X^{(N)}[k] + \frac{1}{2}X^{(N)}[k+\tfrac{N}{2}]}$$

Note that the second terms in the above equation and the equation for $Y^{(\frac{N}{2})}[k]$ in the first solution are equal because $X^{(N)}[k]$ is periodic with period N.

2. (continued)

Alternate solution: DTFT Approach

Since $x[n]$ is an N point signal,

$$X^{(N)}[k] = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = \mathrm{DTFT}\{x[n]\}\Big|_{\omega=\frac{2\pi k}{N}}$$

Since $y[n]$ is an $N/2$ point signal,

$$Y^{(\frac{N}{2})}[k] = Y(\omega)\Big|_{\omega=\frac{2\pi k}{N/2}} = \mathrm{DTFT}\{y[n]\}\Big|_{\omega=\frac{2\pi k}{N/2}}$$

$y[n]$ is $x[n]$ downsampled by 2.

$$\underset{X(\omega)}{x[n]} \longrightarrow \boxed{2\downarrow} \longrightarrow \underset{Y(\omega) = \frac{1}{2}\sum_{\ell=0}^{1} X\left(\frac{\omega-2\pi\ell}{2}\right)}{y[n]}$$

$$Y(\omega) = \frac{1}{2}X\left(\frac{\omega}{2}\right) + \frac{1}{2}X\left(\frac{\omega}{2}-\pi\right)$$

$$Y^{(\frac{N}{2})}[k] = Y(\omega)\Big|_{\omega=\frac{2\pi k}{N/2}} = Y(\omega)\Big|_{\frac{\omega}{2}=\frac{2\pi k}{N}}$$

$$= \frac{1}{2}X\left(\frac{\omega}{2}\right)\Big|_{\frac{\omega}{2}=\frac{2\pi k}{N}} + \frac{1}{2}X\left(\frac{\omega}{2}-\pi\right)\Big|_{\frac{\omega}{2}=\frac{2\pi k}{N} \Rightarrow \frac{\omega}{2}-\pi=\frac{2\pi(k-\frac{N}{2})}{N}}$$

$$\Rightarrow \boxed{Y^{(\frac{N}{2})}[k] = \frac{1}{2}X^{(N)}[k] + \frac{1}{2}X^{(N)}\left[k-\frac{N}{2}\right]}$$

3. (25) The signal $x(t) = \cos(2\pi(110)t)$ is sampled 16 times at a 320 Hz rate to obtain $x[n], n = 0,\ldots,15$. We then compute the 16-point DFT $X^{(16)}[k]$ of this signal.

   a. (7) Determine the values of k corresponding to the peaks that we would observe in the DFT $X^{(16)}[k]$.

   b. (2) Are picket fence and leakage present in this case?

   c. (10) Find an expression for the DFT $X^{(16)}[k]$. You may use anything from the formula sheet "ECE 438 Essential Definitions and Relations" to solve this problem.

   d. (6) Sketch the DFT $X^{(16)}[k]$.

$$x[n] = \cos\left(\frac{2\pi(110)n}{320}\right) = \cos\left(\frac{2\pi n}{16}\cdot \underbrace{\frac{11}{2}}_{k_0}\right)$$

a) Since k_0 is not an integer, leakage occurs. The highest peaks are near $k_0 = 5.5$ and $16 - k_0 = 10.5$.

$\boxed{\text{The values of } k \text{ corresponding to the highest peaks are } 5, 6, 10, 11.}$

b) $\boxed{\text{Yes.}}$ (See part a.)

c) $x[n] = \cos\left(2\pi\left(\frac{11}{32}\right)n\right)$

$$X(\omega) \triangleq \text{DTFT}\{x[n]\} = \pi\, \text{rep}_{2\pi}\left[\delta\left(\omega - \frac{2\pi 11}{32}\right) + \delta\left(\omega + \frac{2\pi 11}{32}\right)\right]$$

Let $w[n] = u[n] - u[n-16]$

$$W(\omega) \triangleq \text{DTFT}\{w[n]\} = \frac{\sin\left(\frac{\omega 16}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega\frac{15}{2}}$$

3. (continued)

By the product rule,

$$\text{DTFT}\{x[n]\cdot w[n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mu)\, W(\omega-\mu)\, d\mu$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi\left[\delta\left(\mu - \frac{2\pi 11}{32}\right) + \delta\left(\mu + \frac{2\pi 11}{32}\right)\right] W(\omega-\mu)\, d\mu$$

$$= \frac{1}{2}\left[W\left(\omega - \frac{2\pi 11}{32}\right) + W\left(\omega + \frac{2\pi 11}{32}\right)\right]$$

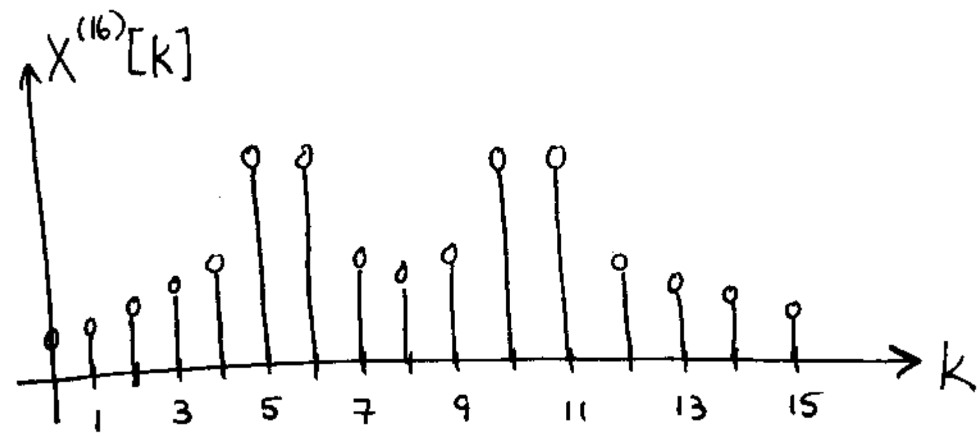
Using the relationship between the DTFT and DFT:

$$X^{(16)}[k] = \text{DTFT}\{x[n]\, w[n]\}\Big|_{w = \frac{2\pi k}{16}}$$

$$= \frac{1}{2}\left[W\left(\frac{2\pi}{16}\left(k - \frac{11}{2}\right)\right) + W\left(\frac{2\pi}{16}\left(k + \frac{11}{2}\right)\right)\right]$$

$$\boxed{X^{(16)}[k] = \frac{1}{2}\left[\frac{\sin\left(\pi\left(k - \frac{11}{2}\right)\right)}{\sin\left(\frac{\pi}{16}\left(k - \frac{11}{2}\right)\right)}\, e^{-j\frac{15\pi}{16}\left(k - \frac{11}{2}\right)} + \frac{\sin\left(\pi\left(k + \frac{11}{2}\right)\right)}{\sin\left(\frac{\pi}{16}\left(k + \frac{11}{2}\right)\right)}\, e^{-j\frac{15}{16}\pi\left(k + \frac{11}{2}\right)}\right]}$$

d)



4. (25 pts) You have a subroutine for the N-point radix-2 FFT where N is any power of 2. You wish to compute an exact 24-point DFT.

a) (10) Derive a set of equations that shows how the 24-point DFT can be efficiently calculated by using your radix-2 FFT subroutine.

b) (9) Draw a block diagram for your 24-point FFT algorithm. Do not show any internal details for the radix-2 part of this algorithm. It should be treated as a black box.

c) (2) Find the approximate number of complex operations (each complex operation consists of one complex multiplication and one complex addition) required to directly compute the 24-point DFT.

d) (4) Find the approximate number of complex operations required to compute the 24-point DFT using your 24-point FFT algorithm.

a) $24 = 8 \cdot 3 = 2^3 \cdot 3$

Divide the 24^{pt} DFT into 3 8-pt. DFTs.

$$X^{(24)}[k] = \sum_{m=0}^{2} \sum_{n=0}^{7} x[3n+m] e^{-j\frac{2\pi k(3n+m)}{24}}$$

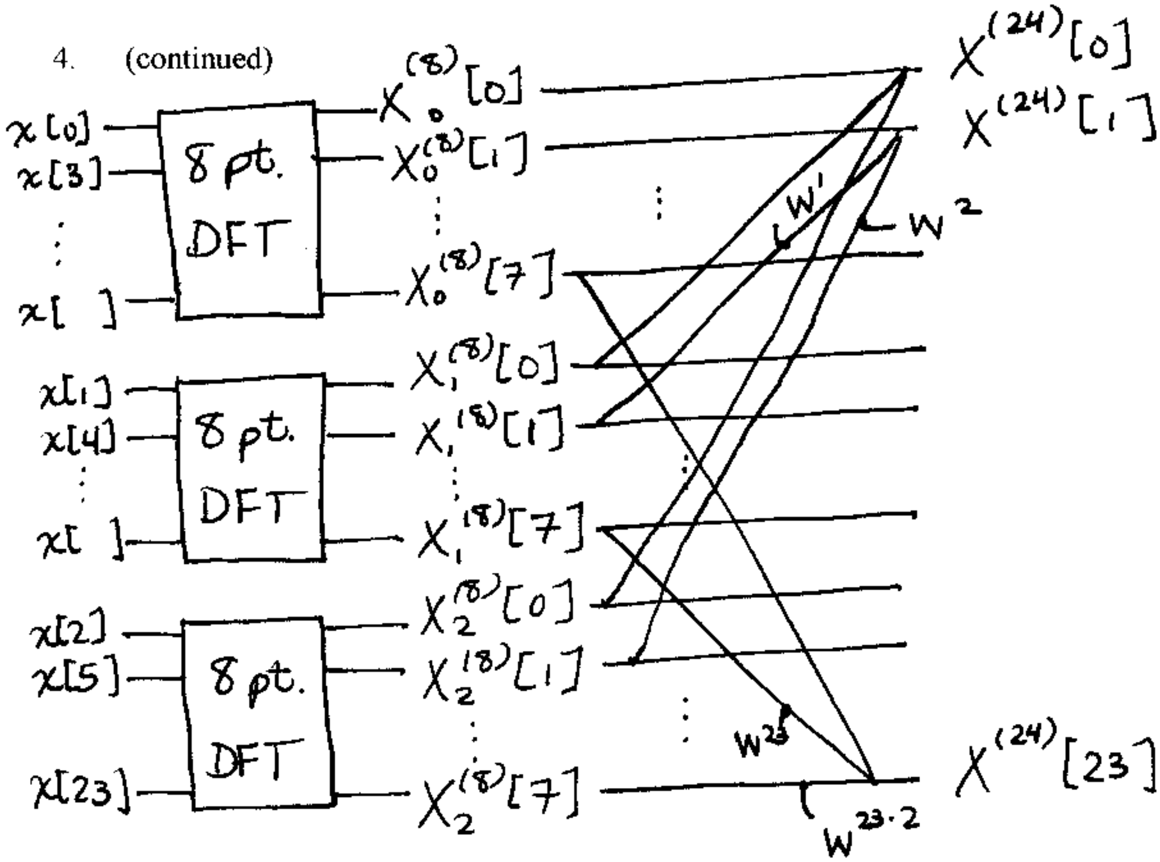
$$= \sum_{m=0}^{2} e^{-j\frac{2\pi k m}{24}} \underbrace{\sum_{n=0}^{7} x[3n+m] e^{-j\frac{2\pi k n}{8}}}_{X_m^{(8)}[k]}$$

$$= X_0^{(8)}[k] + e^{-j\frac{2\pi k}{24}} X_1^{(8)}[k] + e^{-j\frac{2\pi k}{12}} X_2^{(8)}[k] \quad (*)$$

Use the subroutine to compute $X_m^{(8)}[k]$, $m=0,1,2$, then use $(*)$ to get $X^{(24)}[k]$.

4. (continued)

b)



where $W = e^{-j\frac{2\pi}{24}}$

c) $24^2 = \boxed{576}$ Complex Operations

d) Each 8-pt. DFT: $8 \log_2 8 = 8 \cdot 3 = 24$ COs.

To get from $X_m^{(8)}[k]$ to $X^{(24)}[k]$, we need 2 COs per k.

Total COs: $3 \cdot 24 + 24 \cdot 2 = \boxed{120}$