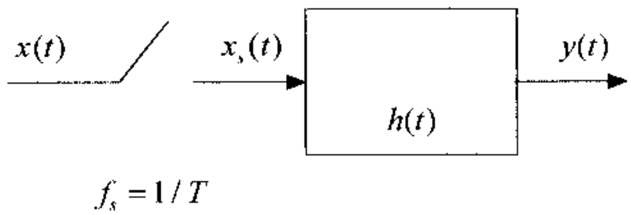
17 February 2004 **Name:** Solution

EE 438 **Exam No. 1** **Spring 2004**

- You have 70 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the system shown below:



The impulse response of the filter is given by $h(t) = rect\left(\frac{t - T/2}{T}\right)$. The ideal sampler can be represented by $x_s(t) = \text{comb}_T[x(t)]$.

a. (9) Find a simple expression for the CTFT $Y(f)$ of the output $y(t)$ in terms of the CTFT $X(f)$ of the input $x(t)$. Your final answer should not contain any operators such as rep or comb.

Suppose that T = 0.01 sec and $x(t) = \text{sinc}(t/0.04)$.

b. (8) Carefully sketch both $x(t)$ and $y(t)$. Be sure to label both axes, and to accurately dimension the features of the signal.

c. (8) Carefully sketch both $X(f)$ and $Y(f)$. Be sure to label both axes, and to accurately dimension the features of the spectrum.

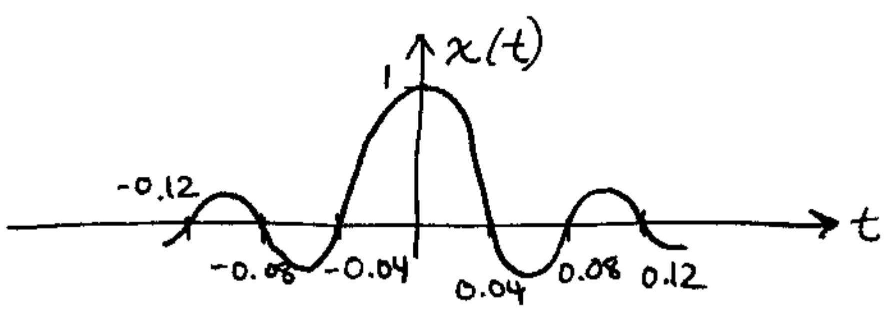
a) $X_s(f) = \frac{1}{T} rep_{\frac{1}{T}}[X(f)] = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f - \frac{k}{T})$

$H(f) = T\,\text{sinc}(Tf)e^{-j2\pi f\frac{T}{2}}$

$Y(f) = X_s(f)\,H(f)$

$= \text{sinc}(Tf)e^{-j2\pi f\frac{T}{2}}\sum_{k=-\infty}^{\infty} X(f - \frac{k}{T})$

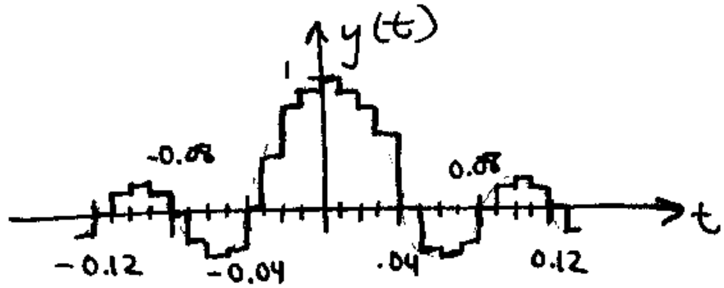
b)



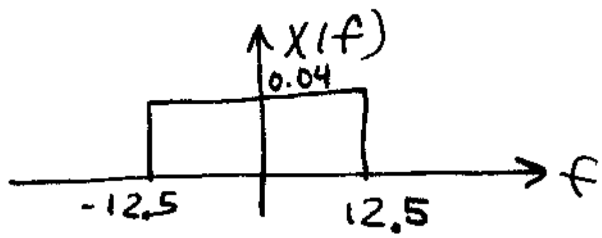
1. (continued)

b) (cont.)

$$y(t) = \text{comb}_T[x(t)] * h(t)$$
$$= \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT) * \text{rect}\left(\frac{t-T/2}{T}\right)$$
$$= \sum_{k=-\infty}^{\infty} x(kT)\,\text{rect}\left(\frac{t-\frac{T}{2}-kT}{T}\right)$$

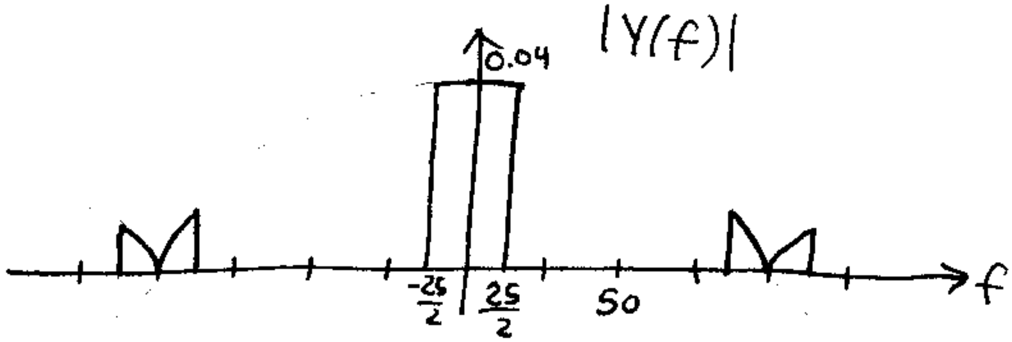


c) $X(f) = 0.04\,\text{rect}(0.04 f)$



$\frac{1}{0.04} = 25$

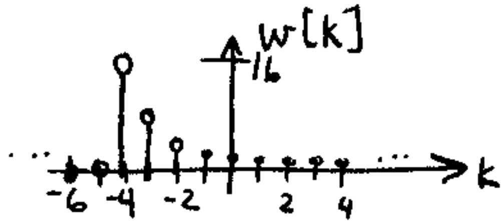
$$Y(f) = \text{sinc}(0.01f)\,e^{-j\pi f 0.01} \sum_{k=-\infty}^{\infty} 0.04\,\text{rect}(0.04(f-100k))$$

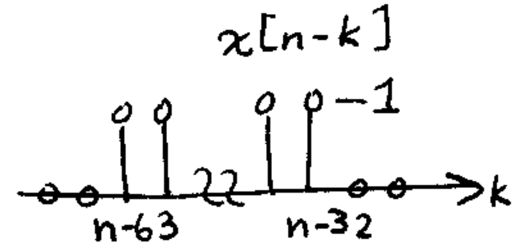


2. (25 pts.) Find the convolution $y[n] = w[n] * x[n]$ of the signals $w[n]$ and $x[n]$ given below:

$$w[n] = 2^{-n} u[n+4]$$

$$x[n] = \begin{cases} 1, & 32 \le n \le 63 \\ 0, & \text{else} \end{cases}$$





$n - 32 < -4 \Rightarrow n < 28$:

$$y[n] = 0$$

$n - 32 \ge -4$ and $n - 63 \le -4 \Rightarrow 28 \le n \le 59$:

$$y[n] = \sum_{k=-4}^{n-32} 2^{-k} = 2^4 \sum_{k=0}^{n-28} 2^{-k} = 2^4 \sum_{k=0}^{n-28} \left(\tfrac{1}{2}\right)^k$$

$$= 2^4 \, \frac{1 - \left(\frac{1}{2}\right)^{n-27}}{1 - \frac{1}{2}} = 2^5 - \left(\tfrac{1}{2}\right)^{n-32}$$

$n - 63 > -4 \Rightarrow 59 < n$:

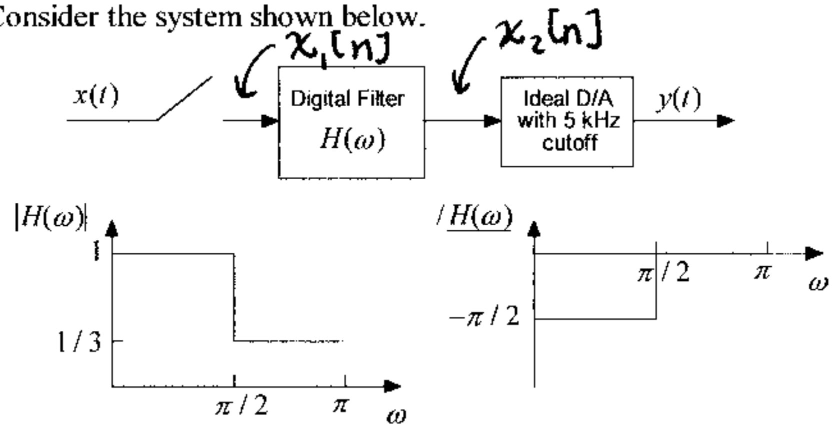
$$y[n] = \sum_{k=n-63}^{n-32} 2^{-k} = 2^{-n+63} \sum_{k=0}^{31} 2^{-k} = 2^{-n+63} \, \frac{1 - \left(\frac{1}{2}\right)^{32}}{1 - \frac{1}{2}}$$

$$= 2^{-n+64} - 2^{-n+32} = 2^{-n}\left(2^{64} - 2^{32}\right)$$

Summary:

$$y[n] = \begin{cases} 0 & , \; n < 28 \\ 2^5 - \left(\frac{1}{2}\right)^{n-32} & , \; 28 \le n \le 59 \\ 2^{-n}\left(2^{64} - 2^{32}\right) & , \; 59 < n \end{cases}$$

3. (25) Consider the system shown below.



The input signal is $x(t) = \cos(2\pi(1000)t) + 10\cos(2\pi(7000)t$. Find the output $y(t)$.

Note: you should assume that $|H(\omega)|$ is an even function of ω; and $\underline{/H(\omega)}$ is an odd function of ω.

$$X(f) = \frac{1}{2}\{\delta(f-1000) + \delta(f+1000)\} + 5\{\delta(f-7000) + \delta(f+7000)\}$$

$$X_1(\omega) = 10k \; rep_{2\pi}\left[X(f)\Big|_{f=\frac{\omega}{2\pi/f_s}}\right]$$

$$= 10k \; rep_{2\pi}\left[\frac{1}{2}\left\{\delta\left(\frac{\omega - \pi/5}{2\pi/10k}\right) + \delta\left(\frac{\omega + \pi/5}{2\pi/10k}\right)\right\} + \underbrace{5\left\{\delta\left(\frac{\omega - 7\pi/5}{2\pi/10k}\right) + \delta\left(\frac{\omega + 7\pi/5}{2\pi/10k}\right)\right\}}_{\text{These impulses cause aliasing. Add and subtract } 2\pi \text{ to find their location in the range } |\omega| < \pi.}\right]$$

These impulses cause aliasing. Add and subtract 2π to find their location in the range $|\omega| < \pi$.

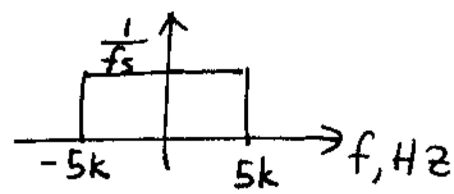
$$= 10k \; rep_{2\pi}\left[\frac{1}{2}\left\{\delta\left(\frac{\omega - \pi/5}{2\pi/10k}\right) + \delta\left(\frac{\omega + \pi/5}{2\pi/10k}\right)\right\} + 5\left\{\delta\left(\frac{\omega - 3\pi/5}{2\pi/10k}\right) + \delta\left(\frac{\omega + 3\pi/5}{2\pi/10k}\right)\right\}\right]$$

3. (continued)

$H(\omega)$ changes the phase of the first two impulses and the magnitude of the third and fourth impulses.

$$X_2(\omega) = 10k \; rep_{2\pi}\Big[\tfrac{1}{2}\Big\{ \delta\Big(\frac{\omega - \pi/5}{2\pi/10k}\Big) e^{-j\pi/2} + \delta\Big(\frac{\omega + \pi/5}{2\pi/10k}\Big) e^{j\pi/2} \Big\} + \tfrac{5}{3}\Big\{ \delta\Big(\frac{\omega - 3\pi/5}{2\pi/10k}\Big) + \delta\Big(\frac{\omega + 3\pi/5}{2\pi/10k}\Big) \Big\} \Big]$$

$$Y(f) = X_2(\omega)\Big|_{\omega = \frac{2\pi f}{10k}} \cdot$$



$$= \frac{1}{2}\Big\{ \delta(f - 1000) e^{-j\pi/2} + \delta(f + 1000) e^{j\pi/2} \Big\} + \frac{5}{3}\Big\{ \delta(f - 3000) + \delta(f + 3000) \Big\}$$

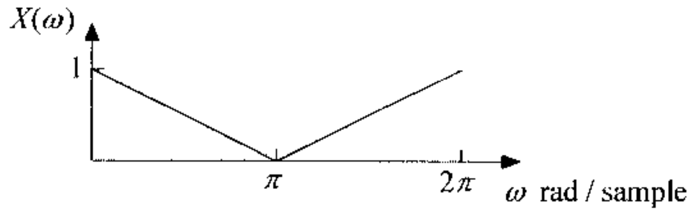
$$y(t) = \frac{1}{2}\Big\{ e^{j(2\pi 1000 t - \pi/2)} + e^{-j(2\pi 1000 t - \pi/2)} \Big\} + \frac{5}{3}\Big\{ e^{j 2\pi 3000 t} + e^{-j 2\pi 3000 t} \Big\}$$

$$= \cos(2\pi 1000 t - \pi/2) + \frac{10}{3}\cos(2\pi 3000 t)$$

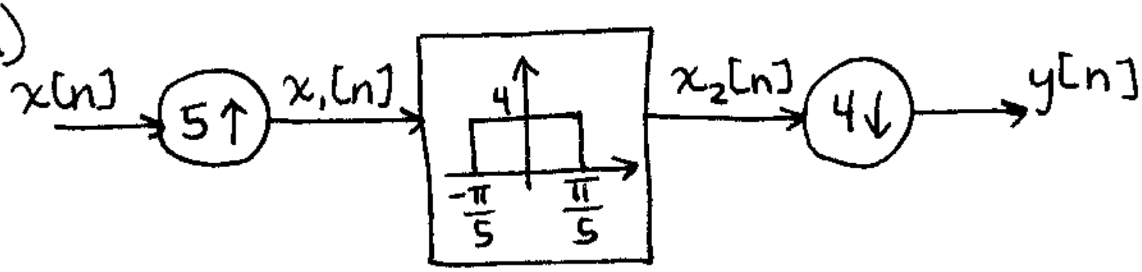
4. (25 pts) Using an appropriate combination of downsamplers, upsamplers, and digital filters, design a system to increase the effective sampling rate by a factor of 1.25. The input to your system will be a digital signal *x[n]*; and the output will be a digital signal *y[n]* effectively sampled at a rate that is a factor of 1.25 greater than the sampling rate for *x[n]*.

   a) (13) Draw a complete block diagram of your system. Be sure to specify all digital filter frequency responses and upsampler/downsampler factors "D".

   b) (12) Suppose the input *x[n]* to your system has the DTFT $X(\omega)$ shown below. Sketch the DTFTs of the signals at every point in your system, including the DTFT $Y(\omega)$ of the output *y[n]*.



a)



b)