16 February 2004 **Name:** Solution

EE 438 **Exam No. 1** **Spring 2004**

- You have 60 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the signal

$$x(t) = \begin{cases} \cos(2\pi t/4), & |t| < 1 \\ 0, & \text{else} \end{cases}$$

a. (3) Sketch $x(t)$. Be sure to label both axes, and to accurately dimension the features of the signal.

b. (7) Use standard functions and transform relations to find the CTFT $X(f)$ of $x(t)$. Do not evaluate the Fourier integral directly. If you find $X(f)$ in this manner, you will not receive full credit.

c. (3) Sketch $X(f)$. Be sure to label both axes, and to accurately dimension the features of the spectrum.

Now consider the signal

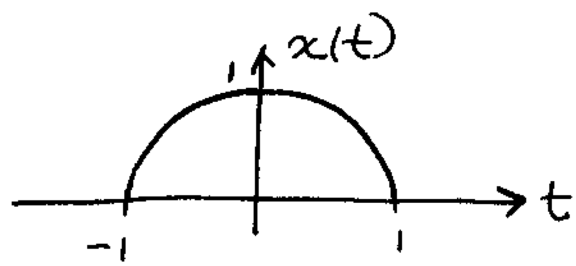
$$y(t) = \text{rep}_4[x(t)]$$

d. (3) Sketch $y(t)$. Be sure to label both axes, and to accurately dimension the features of the signal.

e. (6) Use standard functions and transform relations to find the CTFT $Y(f)$ of $y(t)$. Be sure to remove all operators from your final answer.

f. (3) Sketch $Y(f)$. Be sure to label both axes, and to accurately dimension the features of the spectrum.

a.



b. $x(t) = \cos\left(\frac{2\pi t}{4}\right) \text{rect}\left(\frac{t}{2}\right)$

$$\cos\left(\frac{2\pi t}{4}\right) \xleftrightarrow{CTFT} \frac{1}{2}\left[\delta(f - \tfrac{1}{4}) + \delta(f + \tfrac{1}{4})\right]$$

$$\text{rect}\left(\frac{t}{2}\right) \longleftrightarrow 2\,\text{sinc}(2f)$$

$$a(t) \cdot b(t) \longleftrightarrow A(f) * B(f)$$

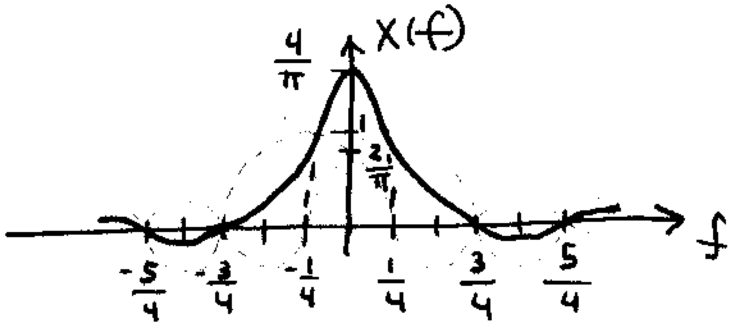
1. (continued)

b. (cont.)

$$X(f) = \frac{1}{2}\left[\delta\left(f-\tfrac{1}{4}\right)+\delta\left(f+\tfrac{1}{4}\right)\right] * 2\,\mathrm{sinc}(2f)$$

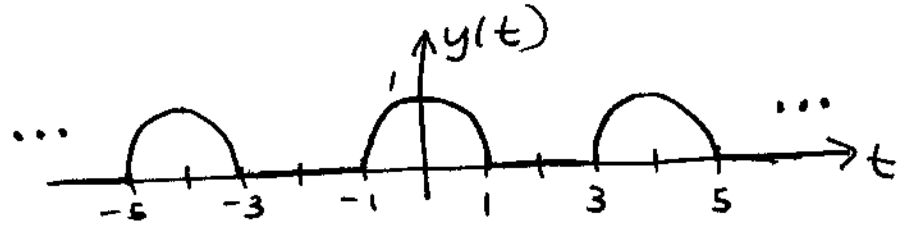
$$\boxed{X(f) = \mathrm{sinc}\left(2\left(f-\tfrac{1}{4}\right)\right) + \mathrm{sinc}\left(2\left(f+\tfrac{1}{4}\right)\right)}$$

c.



$$\mathrm{sinc}\left(\tfrac{1}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$$

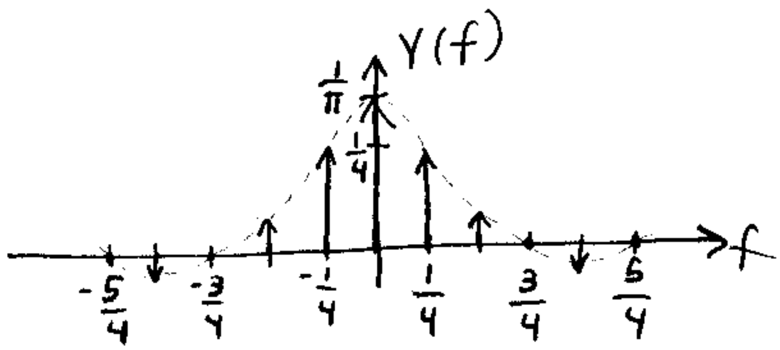
d.



e. $Y(f) = \frac{1}{4}\,\mathrm{comb}_{\frac{1}{4}}[X(f)]$

$$= \frac{1}{4}\sum_{k=-\infty}^{\infty}\left[\mathrm{sinc}\left(\tfrac{1}{2}(k-1)\right)+\mathrm{sinc}\left(\tfrac{1}{2}(k+1)\right)\right]\delta\left(f-\tfrac{k}{4}\right)$$

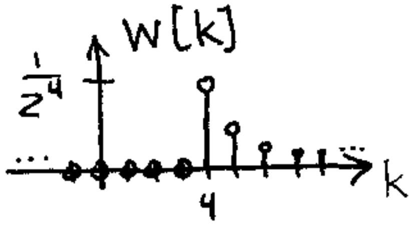
f.

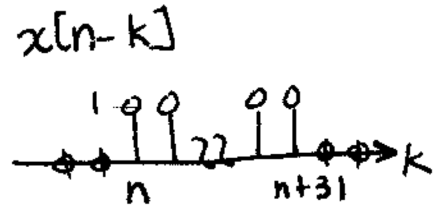


2. (25 pts.) Find the convolution $y[n] = w[n] * x[n]$ of the signals $w[n]$ and $x[n]$ given below:

$$w[n] = 2^{-n} u[n-4]$$

$$x[n] = \begin{cases} 1, & -31 \le n \le 0 \\ 0, & \text{else} \end{cases}$$





$n+31 < 4 \Rightarrow n < -27$:

$$y[n] = 0$$

$n+31 \ge 4$ and $n < 4 \Rightarrow -27 \le n < 4$:

$$y[n] = \sum_{k=4}^{n+31} 2^{-k} = 2^{-4} \sum_{k=0}^{n+27} 2^{-k} = \left(\tfrac{1}{2}\right)^4 \sum_{k=0}^{n+27} \left(\tfrac{1}{2}\right)^k$$

$$= \left(\tfrac{1}{2}\right)^4 \cdot \frac{1-\left(\frac{1}{2}\right)^{n+28}}{1-\frac{1}{2}} = \left(\tfrac{1}{2}\right)^3 - \left(\tfrac{1}{2}\right)^{n+31}$$

$n \ge 4$:

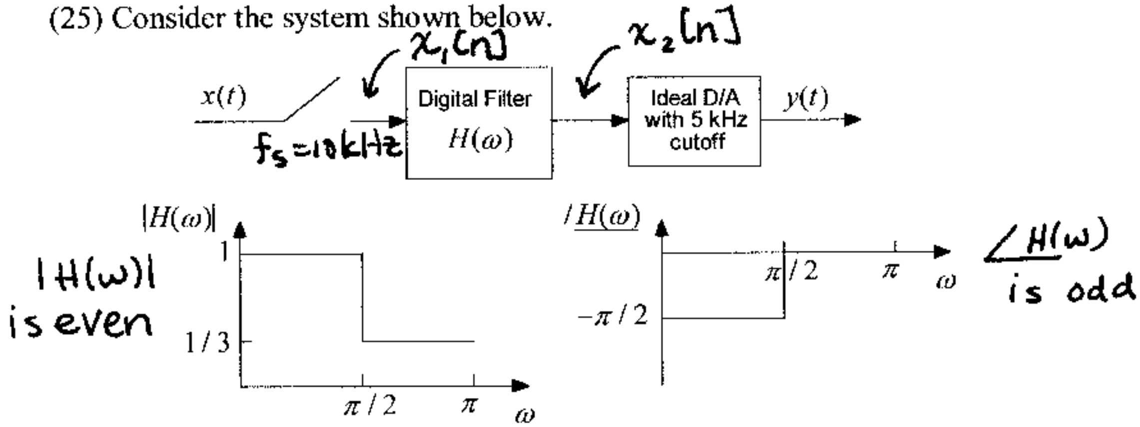
$$y[n] = \sum_{k=n}^{n+31} 2^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{31} \left(\tfrac{1}{2}\right)^k$$

$$= \left(\tfrac{1}{2}\right)^n \frac{1-\left(\frac{1}{2}\right)^{32}}{1-\frac{1}{2}} = \left(\tfrac{1}{2}\right)^{n-1} - \left(\tfrac{1}{2}\right)^{n+31}$$

Summary:

$$y[n] = \begin{cases} 0, & n < -27 \\ \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^{n+31}, & -27 \le n < 4 \\ \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{n+31}, & 4 \le n \end{cases}$$

3. (25) Consider the system shown below.



The input signal is $x(t) = \cos(2\pi(2000)t) + 10\cos(2\pi(6000)t$. Find the output $y(t)$.

$$X(f) = \frac{1}{2}\{\delta(f-2k)+\delta(f+2k)\} + 5\{\delta(f-6k) + \delta(f+6k)\}$$

$$X_1(\omega) = 10k \ \text{rep}_{2\pi}[X(f)]\Big|_{f=\frac{\omega}{2\pi/f_s}}$$

$$= 10k \ \text{rep}_{2\pi}\Big[\frac{1}{2}\Big\{\delta\Big(\frac{\omega - 2\pi/5}{2\pi/10k}\Big) + \delta\Big(\frac{\omega + 2\pi/5}{2\pi/10k}\Big)\Big\}$$

$$+ 5\Big\{\underbrace{\delta\Big(\frac{\omega - 6\pi/5}{2\pi/10k}\Big) + \delta\Big(\frac{\omega + 6\pi/5}{2\pi/10k}\Big)}\Big\}\Big]$$

Since these impulses are outside $|\omega| < \pi$, the sampler has caused aliasing. Add and subtract 2π to determine where the impulses are located in the range $|\omega| < \pi$.

$$= 10k \ \text{rep}_{2\pi}\Big[\frac{1}{2}\Big\{\delta\Big(\frac{\omega - 2\pi/5}{2\pi/10k}\Big) + \delta\Big(\frac{\omega + 2\pi/5}{2\pi/10k}\Big)\Big\}$$

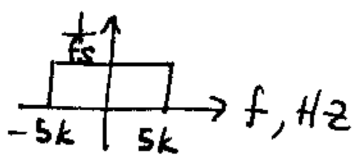
$$+ 5\Big\{\delta\Big(\frac{\omega - 4\pi/5}{2\pi/10k}\Big) + \delta\Big(\frac{\omega + 4\pi/5}{2\pi/10k}\Big)\Big\}\Big]$$

3. (continued)

$H(\omega)$ does not change the magnitude of the first two impulses but it does change the phase. $H(\omega)$ only changes the magnitude of the third and fourth impulses.

$$X_2(\omega) = 10k \; \mathrm{rep}_{2\pi}\left[\frac{1}{2}\left\{ \delta\left(\frac{\omega - 2\pi/5}{2\pi/10k}\right)e^{-j\pi/2} + \delta\left(\frac{\omega + 2\pi/5}{2\pi/10k}\right)e^{j\pi/2} \right\} + \frac{5}{3}\left\{ \delta\left(\frac{\omega - 4\pi/5}{2\pi/10k}\right) + \delta\left(\frac{\omega + 4\pi/5}{2\pi/10k}\right) \right\} \right]$$

$$Y(f) = X_2(\omega)\Big|_{\omega = \frac{2\pi f}{10k}} \cdot$$



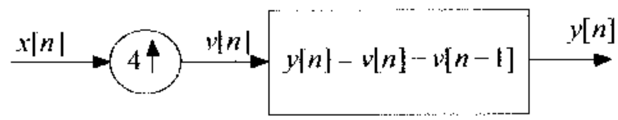
$f_s = 10kHz$

$$= \frac{1}{2}\left\{ \delta(f-2k)e^{-j\pi/2} + \delta(f+2k)e^{j\pi/2} \right\} + \frac{5}{3}\left\{ \delta(f-4k) + \delta(f+4k) \right\}$$

$$y(t) = \frac{1}{2}\left\{ e^{j(2\pi 2kt - \pi/2)} + e^{-j(2\pi 2kt - \pi/2)} \right\} + \frac{5}{3}\left\{ e^{j2\pi 4kt} + e^{-j2\pi 4kt} \right\}$$

$$= \cos(2\pi 2000t - \tfrac{\pi}{2}) + \frac{10}{3}\cos(2\pi 4000t)$$

4. (25 pts) Consider the DT system shown below:



a) (4) Is this system linear? Simply state your answer – "yes" or "no". No justification or proof is needed; and no partial credit will be given.

b) (4) Is this system time-invariant? Simply state your answer – "yes" or "no". No justification or proof is needed; and no partial credit will be given.

c) (10) Find the output $y[n]$ when the input is given by $x[n] = e^{j\omega_0 n}$.

d) (7) Find simple expressions for the magnitude and phase of $y[n]$.

a) Yes.

b) No.

c) $V(\omega) = X(4\omega)$

$$X(\omega) = 2\pi \, rep_{2\pi}[\delta(\omega - \omega_0)]$$

$$V(\omega) = 2\pi \, rep_{2\pi}[\delta(4\omega - \omega_0)]$$

$$= \frac{\pi}{2} rep_{2\pi}\left[\delta\left(\omega - \frac{\omega_0}{4}\right)\right]$$

$$Y(\omega) = V(\omega)[1 - e^{-j\omega}] = V(\omega) e^{-j\frac{\omega}{2}} 2j \sin\left(\frac{\omega}{2}\right)$$

$$= \pi j \sin\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}} rep_{2\pi}\left[\delta\left(\omega - \frac{\omega_0}{4}\right)\right]$$

$$= \pi j \, rep_{2\pi}\left[\sin\left(\frac{\omega_0}{8}\right) e^{-j\frac{\omega_0}{8}} \delta\left(\omega - \frac{\omega_0}{4}\right)\right]$$

$$\boxed{y[n] = \frac{1}{2} j \sin\left(\frac{\omega_0}{8}\right) e^{-j\frac{\omega_0}{8}} e^{j\frac{\omega_0}{4} n}}$$

4. (continued)

d) $|y[n]| = \frac{1}{2}\left|\sin\left(\frac{\omega_0}{8}\right)\right|$

$$\angle y[n] = \frac{\pi}{2} - \frac{\omega_0}{8} + \frac{\omega_0}{4} n - \begin{cases} \pi, & \sin\left(\frac{\omega_0}{8}\right) < 0 \\ 0, & \sin\left(\frac{\omega_0}{8}\right) \geq 0 \end{cases}$$