18 April 2003 **Name:** Solution

EE 438 **Exam No. 3** **Spring 2003**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- Unless explicitly stated to the contrary, you must simplify your answer as much as possible to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider a random signal $X[n]$ with first order probability density

$$f_X(x) = [1 - |x|]\text{rect}(x/2)$$

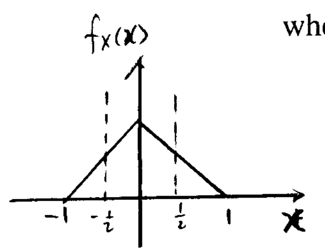
a) (8) Find the mean and variance of X.

b) (7) Suppose we quantize X to just two levels $-\frac{1}{2}$ and $\frac{1}{2}$. Using the approximate analysis developed in class, find the approximate mean-squared quantization error $\hat{e}_{ms}$.

c) (10) Now find the exact mean-squared quantization error e_{ms} for this two level quantizer by evaluating the expression

$$e_{ms} = \int [Q(x) - x]^2 f_X(x)\,dx,$$

where $Q(X)$ is the quantized value of X.



a) $E[X] = \int_{-1}^{1} \underset{\text{odd}}{x} \cdot \underset{\text{even}}{f_X(x)}\,dx = 0$

odd · even = odd

$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = E[X^2] = \int_{-1}^{1} \underbrace{x^2(1-|x|)}_{\text{even}}\,dx$

$= 2\int_0^1 x^2(1-x)\,dx.$

$= 2\int_0^1 x^2 dx - 2\int_0^1 x^3 dx = \frac{2}{3}x^3\Big|_0^1 - 2\cdot\frac{1}{4}x^4\Big|_0^1$

$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

b) $\because$ error $\overset{\text{Approx}}{\sim}$ Uniform$[-\frac{\Delta}{2}, \frac{\Delta}{2}]$

$\therefore \hat{e}_{ms} = \frac{\Delta^2}{12}$ since $\Delta = 1$ in our case $\Big\} \Rightarrow \hat{e}_{ms} = \frac{1}{12}$

1. (continued)

c).

$$e_{ms} = \int (Q(x) - x)^2 f_X(x)\,dx$$

$$= \int_{-1}^{0} (-\tfrac{1}{2} - x)^2 (1+x)\,dx + \int_0^1 (\tfrac{1}{2} - x)^2 (1-x)\,dx$$

$$= 2\int_0^1 (\tfrac{1}{2} - x)^2 (1-x)\,dx$$

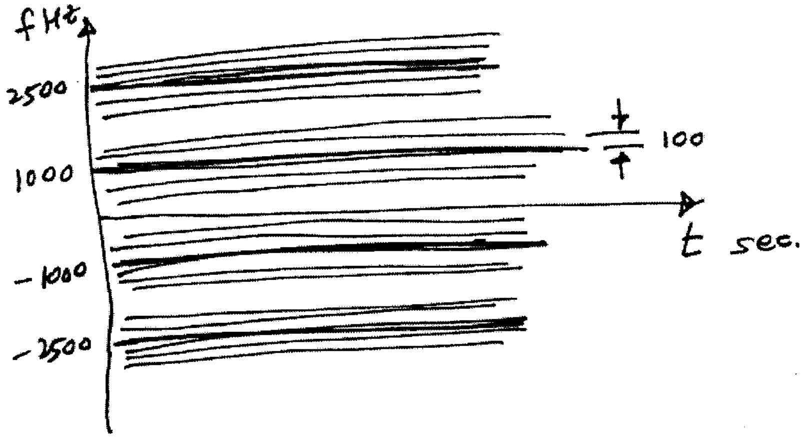
$$t = \tfrac{1}{2} - x \qquad = \quad 2\int_{-\frac{1}{2}}^{\frac{1}{2}} t^2 (\tfrac{1}{2} + t)\,dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} t^2\,dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} 2t^3\,dt$$

$$= \tfrac{1}{3} t^3 \Big|_{-\frac{1}{2}}^{\frac{1}{2}} + 0$$

$$= \tfrac{1}{3} \times \tfrac{1}{8} \times 2 = \frac{1}{12}$$

2. (25 pts.) Consider the continuous-time spectrogram shown below



a) (5) What is the pitch period for this signal?

b) (5) What are the formant frequencies?

c) (5) Is this spectrogram wideband or narrowband?

d) (10) Sketch what the other kind of spectrogram would look like. Be sure to label all important quantities.

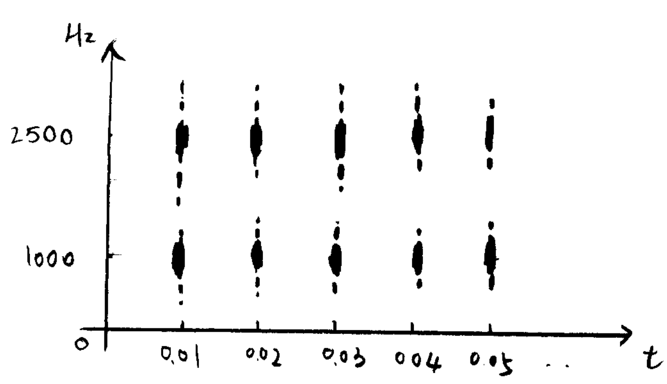
a) $T_p = \frac{1}{\Delta f} = \frac{1}{100} = 0.01$ sec

b) $f_1 = 1000$ Hz

$f_2 = 2500$ Hz

c) narrow band spectrogram (high resolution in frequency!

d)



3. (25 pts) The discrete-time short-time Fourier transform (STDTFT) of a signal $s[n]$ is defined as

$$S(\omega,n)=\sum_k s[k]w[n-k]e^{-j\omega k},$$

where $w[n]$ is the window function, which has DTFT $W(\omega)$.

a) (10) Find the STDTFT $R(\omega,n)$ of the signal $r[n]=s[n-n_0]$, where n_0 is a fixed parameter, in terms of the STDTFT $S(\omega,n)$ and n_0.

b) (5) Find the STDTFT $R(\omega,n)$ of the signal $r[n]=s[n]e^{j\omega_0 n}$, where ω_0 is a fixed parameter, in terms of the STDTFT $S(\omega,n)$ and ω_0.

c) (10) Find the STDTFT $R(\omega,n)$ of the signal $r[n]=s[n]*h[n]$, where $h[n]$ is a the impulse response of a filter (not to be confused with the window $w[n]$), in terms of the STDTFT $S(\omega,n)$, $h[n]$, and the convolution operator $*$.

a). $$R(\omega,n)=\sum_k r[k]w[n-k]e^{-j\omega k} = \sum_k s[k-n_0]w[n-k]e^{-j\omega k}$$

$$= \sum_{k'=k-n_0} s[k']w[n-(k'+n_0)]e^{-j\omega(k'+n_0)}$$

$$= \left(\sum_{k'} s[k']w[\underline{(n-n_0)}-k']e^{-j\omega k'}\right)\cdot e^{-j\omega n_0}$$

$$= S(\omega, n-n_0)\,e^{-j\omega n_0}$$

b). $$R(\omega,n)=\sum_k r[k]w[n-k]e^{-j\omega k} = \sum_k s[k]e^{j\omega_0 k}w[n-k]e^{-j\omega k}$$

$$= \sum_k s[k]w[n-k]e^{-j(\omega-\omega_0)k}$$

$$= S(\omega-\omega_0, n)$$

c). $$R(\omega,n)=\sum_k r[k]w[n-k]e^{-j\omega k} = \sum_k r[k]\,\underbrace{w[n-k]e^{j\omega(n-k)}}\cdot e^{-j\omega n}$$

Solution 1 :

$$= \left(r[n]*\left(w[n]e^{j\omega n}\right)\right)\cdot e^{-j\omega n}$$

$$= \left(h[n]*\underbrace{s[n]*\left(w[n]e^{j\omega n}\right)}\right)\cdot e^{-j\omega n}$$

$$= \left(h[n]*\left(S(\omega,n)\cdot e^{j\omega n}\right)\right)\cdot e^{-j\omega n}$$

Solution 2 , next page.

3. (continued)

Solution 2 :

$$R(\omega,n) = \sum_k r[k] w[n-k] e^{-j\omega k}$$

$$= \sum_k \left(s[k] * h[k]\right) w[n-k] e^{-j\omega k}$$

$$= \sum_k \left(\sum_l h[l]\, s[k-l]\right) w[n-k] e^{-j\omega k}$$

$$= \sum_l h[l] \sum_k s[k-l] w[n-k] e^{-j\omega k}$$

$$\overset{\substack{m=k-l \\ \Downarrow \\ k=m+l}}{=} \sum_l h[l] \sum_m s[m] w[n-l-m] e^{-j\omega(m+l)}$$

$$= \sum_l h[l] e^{-j\omega l} \left(\underline{\sum_m s[m] w[n-l-m] e^{-j\omega m}}\right)$$

$$= \sum_l h[l] e^{-j\omega l} S(\omega, n-l) \qquad (*)$$

From here we can get two equivelant results

1) : $(*) = h_\omega[n] * S(\omega, n)$

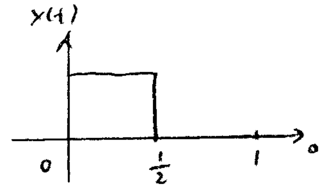
$$h_\omega[n] \triangleq h[n] e^{-j\omega n}$$

2) : $(*) = \left(\sum_l h[l] \underline{S(\omega, n-l) e^{j\omega(n-l)}}\right) \cdot e^{-j\omega n}$

$$= \left(h[n] * \left(S(\omega,n) e^{j\omega n}\right)\right) \cdot e^{-j\omega n}$$

4. (25 pts) Suppose we wish to approximate the waveform $x(t) = rect((t-0.25)/0.5)$ over the interval $0 \le t \le 1$ by the function $\hat{x}(t) = a_0 + a_1\cos(\pi t) + a_2\cos(3\pi t)$.

a. (18) Find the values for the coefficients that minimize
$$E = \int_0^1 [\hat{x}(t) - x(t)]^2 dt.$$



b. (7) Carefully sketch $x(t)$ and $\hat{x}(t)$ on the same axes.

Note: The frequencies of the cosines in the equation for $\hat{x}(t)$ are not the same as those in Problem 4 of Homework No. 9.

The following identity may be useful: $\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

$$E(a_0\, a_1\, a_2) = \int_0^1 (a_0 + a_1\cos\pi t + a_2\cos 3\pi t - x(t))^2 dt$$

a. To Find optimal a_0, a_1, a_2, take partial derivative of $E(a_0\, a_1\, a_2)$

$$\frac{\partial E}{\partial a_0} = 0 \Rightarrow \int_0^1 (a_0 + a_1\cos\pi t + a_2\cos 3\pi t - x(t))\cdot dt = 0$$

$$\Rightarrow a_0 = \int_0^1 x(t)dt - \int_0^1 a_1\cos\pi t\, dt - \int_0^1 a_2\cos 3\pi t\, dt$$

$$= \frac{1}{2} - 0 - 0 = \frac{1}{2}$$

$$\frac{\partial E}{\partial a_1} = 0 \Rightarrow \int_0^1 (a_0 + a_1\cos\pi t + a_2\cos 3\pi t - x(t))\cos\pi t\, dt = 0$$

$$\Rightarrow \int_0^1 a_0\cos\pi t + \int_0^1 a_1\underbrace{\cos^2\pi t}_{\frac{1+\cos 2\pi t}{2}} dt + \int_0^1 a_2\underbrace{\cos 3\pi t\cos\pi t}_{\frac{\cos 2\pi t + \cos 4\pi t}{2}}\cdot dt = \int_0^1 x(t)\cos\pi t\, dt$$

$$0 + \frac{1}{2}a_1 + 0 = \int_0^{\frac{1}{2}} \cos\pi t\, dt = \frac{\sin\pi t}{\pi}\Big|_0^{\frac{1}{2}} = \frac{1}{\pi}$$

$$\therefore a_1 = \frac{2}{\pi}$$

4. (continued)

$$\frac{\partial E}{\partial a_2} = 0 \Rightarrow \int_0^1 (a_0 + a_1 \cos\pi t + a_2 \cos 3\pi t - x(t)) \cos 3\pi t \, dt = 0$$

$$\Rightarrow \int_0^1 a_0 \cos 3\pi t \, dt + \int_0^1 a_1 \underbrace{\cos\pi t \cos 3\pi t}_{\frac{\cos 2\pi t + \cos 4\pi t}{2}} \, dt + \int_0^1 a_2 \underbrace{\cos^2 3\pi t}_{\frac{1+\cos 6\pi t}{2}} = \int_0^1 x(t) \cos 3\pi t$$

$$\Rightarrow 0 + 0 + \frac{1}{2} a_2 = \int_0^{\frac{1}{2}} \cos 3\pi t = \frac{\sin 3\pi t \Big|_0^{\frac{1}{2}}}{3\pi} = -\frac{1}{3\pi}$$

$$\therefore a_2 = -\frac{2}{3\pi}$$

$$\therefore \hat{a}_0 = \frac{1}{2} \quad \hat{a}_1 = \frac{2}{\pi} \quad \hat{a}_2 = -\frac{2}{3\pi} \qquad \hat{x}(t) = \frac{1}{2} + \frac{2}{\pi}\cos\pi t - \frac{2}{3\pi}\cos 3\pi t$$

b.

t	0	0.25	0.5	0.75	1.0
$\hat{x}(t)$	0.925	1.101	0.500	-0.101	0.075

$0 \le t \le 1$