14 March 2003 **Name:**___________________

EE 438 Solution **Exam No. 2** **Spring 2003**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- Unless explicitly stated to the contrary, you must simplify your answer as much as possible to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider a causal linear time-invariant system with input $x[n] = (\frac{1}{2})^n u[n]$ and output $y[n] = \left[2 - (\frac{1}{2})^n\right]u[n]$.

   a) (13) Find the Z transform $Y(z)$. Be sure to state the region of convergence.

   b) (5) Find the transfer function $H(z)$ for the system.

   c) (2) Is the system bounded-input-bounded-output stable? Explain why or why not.

   d) (5) Find the impulse response $h[n]$ for this system.

a). $Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n\geq 0}^{\infty} [2 - (\frac{1}{2})^n] z^{-n} = \sum_{n\geq 0}^{\infty} 2 \cdot z^{-n} - \sum_{n\geq 0}^{\infty} (\frac{1}{2})^n z^{-n}$

$= \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{1}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}$

ROC : $\{|z^{-1}|<1\} \cap \{|z^{-1}\frac{1}{2}|<1\} \Rightarrow |z|>1$

b). $x[n] = (\frac{1}{2})^n u[n] \rightarrow X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}}$ since the system is causal, ROC is $|z|>1$

c). This system is NOT BIBO stable.
Because ROC of $H(z)$ does not include unit cycle.

d). $H(z) = \frac{1}{1-z^{-1}} \underset{\text{Causal.}}{\Longrightarrow} h[n] = u[n]$

c) can be concluded here as well : <u>LTI system</u>: BIBO stable $\rightleftarrows \sum_{n=-\infty}^{\infty}|h(n)| < \infty$
but in our case $\sum_{n=-\infty}^{\infty}|h[n]| = \infty$. hence it is NOT BIBO stable.

2. (25 pts.) Consider the following 20 point signal

$$x[n]=\begin{cases}1, & n=0,1,\dots,9\\ 0, & n=10,11,\dots,19\end{cases}$$

a) (19) Find a simple expression for the 20-point DFT $X[k]$ of this signal.

b) (6) Carefully sketch $X[k]$.

a). $$X_{20}[k] = \sum_{n=0}^{19} x[n]\, e^{-j\frac{2\pi nk}{20}}$$

$$= \sum_{n=0}^{9} 1\cdot e^{-j\frac{2\pi nk}{20}} = \sum_{n=0}^{9} \left(e^{-j\frac{2\pi k}{20}}\right)^n$$

① $k=0 \Rightarrow e^{-j\frac{2\pi k}{20}} = 1$

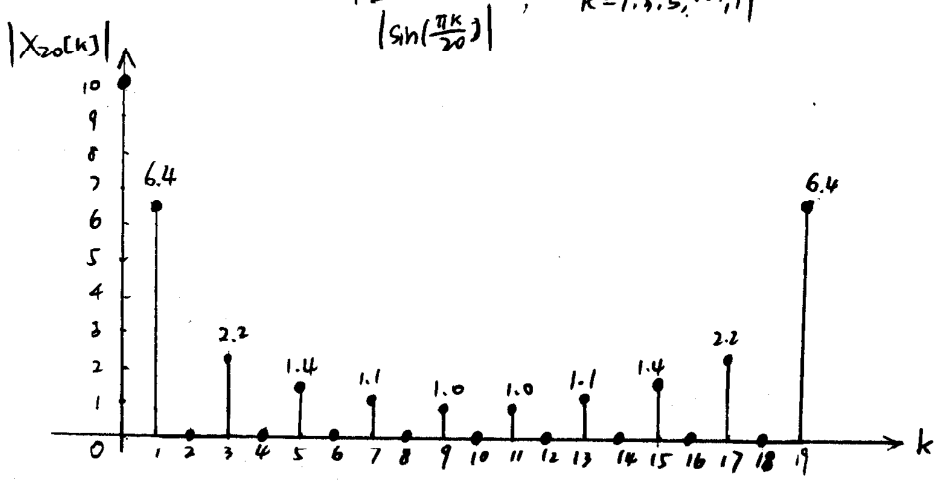
$$X_{20}[0] = \sum_{n=0}^{9} 1^n = \sum_{n=0}^{9} 1 = 10$$

② $k\neq 0,\ k=1,2,\dots,19 \Rightarrow e^{-j\frac{2\pi k}{20}} \neq 1$

$$X_{20}[k] = \frac{1-e^{-j\frac{2\pi k}{20}\cdot 10}}{1-e^{-j\frac{2\pi k}{20}}} = \frac{1-e^{-j\pi k}}{1-e^{-j\frac{2\pi k}{20}}} = \frac{1-(-1)^k}{e^{-j\frac{\pi k}{20}}\left(2j\sin(\frac{\pi k}{20})\right)} = e^{j\frac{\pi k}{20}}\,\frac{1-(-1)^k}{2j\sin(\frac{\pi k}{20})}$$

$$\therefore\ X_{20}[k] = \begin{cases} 10, & k=0\\ 0, & k=2,4,6,\dots,18\\ \dfrac{e^{j\frac{\pi k}{20}}}{j\sin(\frac{\pi k}{20})}, & k=1,3,5,\dots,19\end{cases}$$

b). $$|X_{20}[k]| = \begin{cases} 10 & k=0\\ 0, & k=2,4,6,\dots,18\\ \dfrac{1}{|\sin(\frac{\pi k}{20})|}, & k=1,3,5,\dots,19\end{cases}$$



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3. (25 pts.) You have an integrated circuit chip that computes the radix 2 FFT for length 64 points. You would like to use one or more of these chips to efficiently compute a 192 point DFT by embedding the 64-point FFT chips in an application specific integrated circuit (ASIC).

   a) (10) Derive an expression for the 192 point DFT in terms of the 64-point DFT.

   b) (10) Draw a flow diagram that shows the layout of the ASIC. Be sure to completely label your diagram showing all twiddle factors.

   c) (2) How many 64-point FFT chips will be required?

   d) (3) Compute the required computation to calculate the 192-point DFT in terms of the number of complex operations.

a). $192 = 64 \times 3$

$$X_{192}[k] = \sum_{n=0}^{191} x[n] e^{-j\frac{2\pi kn}{192}} = \sum_{\substack{n=3m\\n=0}}^{189} x[n] e^{-j\frac{2\pi kn}{192}} + \sum_{\substack{n=3m+1\\n=1}}^{190} x[n] e^{-j\frac{2\pi kn}{192}} + \sum_{\substack{n=3m+2\\n=2}}^{191} x[n] e^{-j\frac{2\pi kn}{192}}$$

$\therefore$ Change variable $\displaystyle \sum_{m=0}^{63} x[3m] e^{-j\frac{2\pi k\cdot 3m}{192}} + \sum_{m=0}^{63} x[3m+1] e^{-j\frac{2\pi k(3m+1)}{192}} + \sum_{m=0}^{63} x[3m+2] e^{-j\frac{2\pi k(3m+2)}{192}}$

$$= \sum_{m=0}^{64} x[3m] e^{-j\frac{2\pi km}{64}} + e^{-j\frac{2\pi k}{192}} \sum_{m=0}^{63} x[3m+1] e^{-j\frac{2\pi km}{64}} + e^{-j\frac{4\pi k}{192}} \sum_{m=0}^{63} x[3m+2] e^{-j\frac{2\pi km}{64}}$$

$$= X^0_{64}[k] + W^k_{192}\cdot X^1_{64}[k] + W^{2k}_{192}\cdot X^2_{64}[k]$$

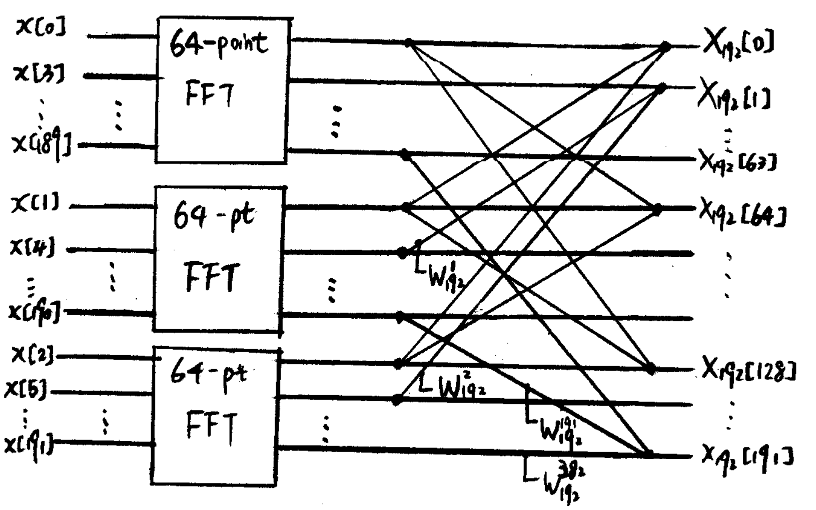
$X^0_{64}[k]$ is the 64-point FFT of $x_0[n] \triangleq \{x[0], x[3], x[6], \cdots, x[189]\}$

$X^1_{64}[k]$ 〃 〃 〃 $x_1[n] \triangleq \{x[1], x[4], x[7], \cdots, x[190]\}$

$X^2_{64}[k]$ is 〃 〃 〃 $x_2[n] \triangleq \{x[2], x[5], x[8], \cdots, x[191]\}$

$W^k_N \triangleq e^{-j\frac{2\pi k}{N}}$ twiddle factor.

b)



c) 3 chips.

d).

3 64-point FFT:

$3 \times (64 \times \log_2 64) = 1152$

twiddle factors:

$2 \times 192 = 384$

Total # of cos:

$1152 + 384 = 1536$

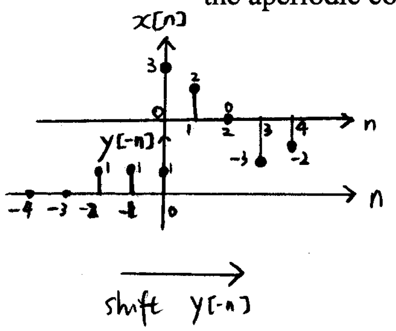
4. (25 pts) Consider the following signals

n	0	1	2	3	4
$x[n]$	3	2	0	−3	−2
$y[n]$	1	1	1	0	0

a) (10) Calculate the aperiodic convolution of $x[n]$ and $y[n]$.

b) (15) Calculate the periodic (circular) convolution of $x[n]$ and $y[n]$ with period 5.

c) (3) To what length must these signals be padded with zeros so that a portion of the periodic convolution will yield the same result as the non-zero part of the aperiodic convolution?

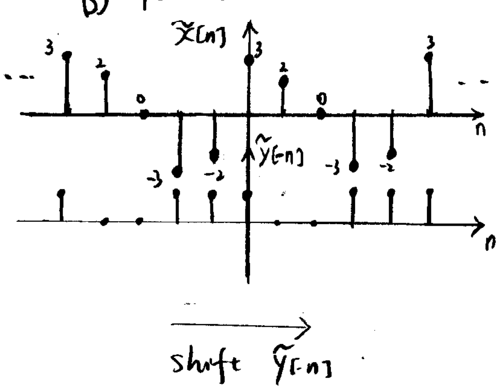


$z[n] \triangleq x[n] * y[n]$ aperiodic convolution

a)

n	...	−1	0	1	2	3	4	5	6	7	8
$z[n]$	...	0	3	5	5	−1	−5	−5	−2	0	...

b) periodic convolution



$\tilde{x}[n], \tilde{y}[n]$ are the periodic extension of $x[n]$ and $y[n]$ respectively, with period 5.

$$\tilde{z}[n] = \tilde{y}[n] * \tilde{x}[n] = \sum_{k=0}^{4} \tilde{x}[k]\tilde{y}[n-k] = \sum_{k=0}^{4} x[k]\,y[(n-k) \bmod 5]$$

$$= \begin{cases} \sum_{k=0}^{n} x[k]y[n-k] + \sum_{k=n+1}^{4} x[k]y[5+n-k] & 0 \le n \le 3 \\ \sum_{k=0}^{4} x[k]y[n-k] & n=4 \end{cases}$$

$$\therefore \quad \tilde{z}[n] = \begin{cases} z[n] + z[n+5], & 0 \le n \le 3 \\ z[n], & n=4 \end{cases}$$

n	0	1	2	3	4
$\tilde{z}[n]$	−2	3	5	−1	−5

c) Non-periodic convolution $z[n]$ is $(5+3-1) = 7$ points long, so must pad $x[n]$ and $y[n]$ to at least 7 points long, i.e.: pad 2 more zeros.