14 February 2003 **Name:** Solution

EE 438 **Exam No. 1** **Spring 2003**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider a linear, time-invariant system defined by the equation

$$y[n] = x[n] - x[n-2] - y[n-1] \qquad (*)$$

a. (12) Find a simple expression for the frequency response $H(\omega)$.

b. (7) Find and sketch the magnitude $|H(\omega)|$. Be sure to label your axes.

c. (6) Find and sketch the phase $\angle H(\omega)$. Be sure to label your axes.

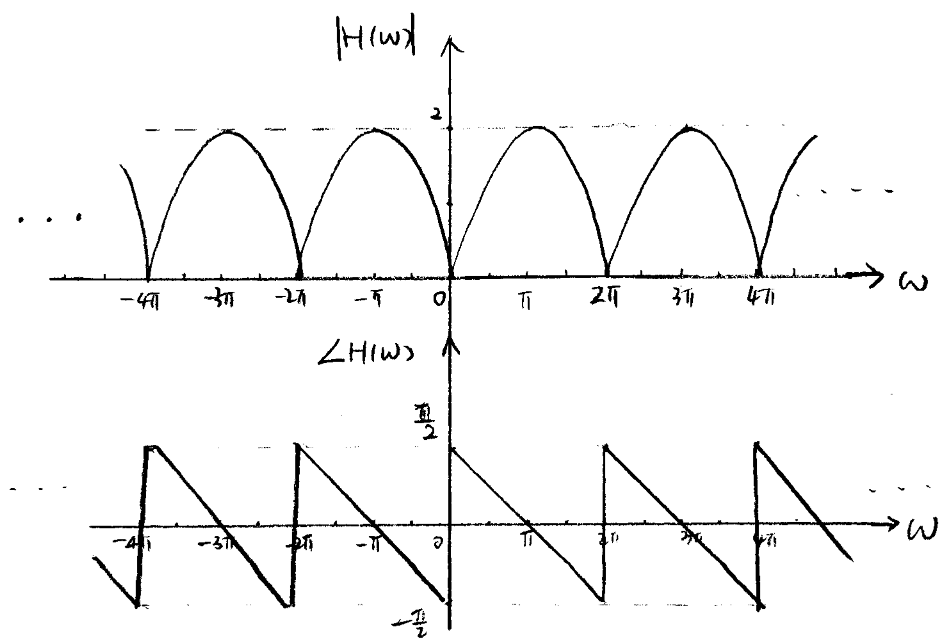
a. Let $x[n] = e^{j\omega n}$, then $y[n] = H(\omega) e^{j\omega n}$ (only valid for complex exponential input)

$y[n-1] = H(\omega) e^{j\omega(n-1)}$ Substitute in $(*)$ $\Rightarrow$ $H(\omega) e^{j\omega n} = e^{j\omega n} - e^{j\omega(n-2)} - H(\omega) e^{j\omega(n-1)}$

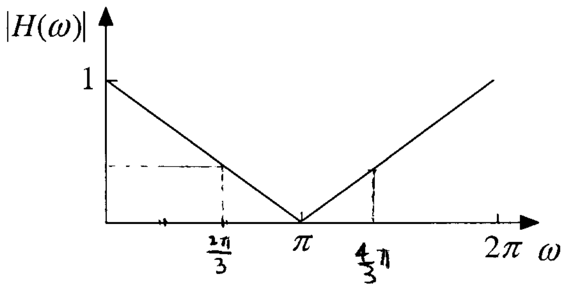
$$\Rightarrow H(\omega) = \frac{1 - e^{-j2\omega}}{1 + e^{-j\omega}} = \frac{(1 - e^{-j\omega})(1 + e^{-j\omega})}{1 + e^{-j\omega}} = 1 - e^{-j\omega} = \underline{\underline{e^{-j\frac{\omega}{2}} \cdot 2j \sin\frac{\omega}{2}}}$$

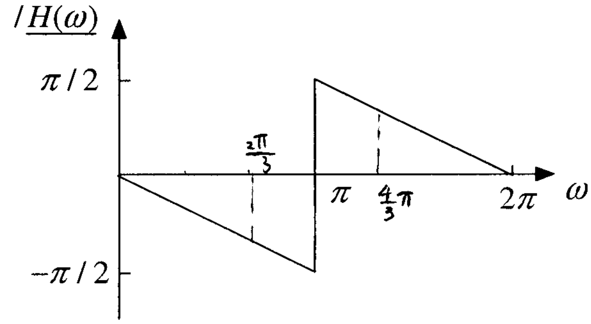
b. $$|H(\omega)| = \left| 2j \sin(\tfrac{\omega}{2}) e^{-j\frac{\omega}{2}} \right| = 2 \left| \sin(\tfrac{\omega}{2}) \right|$$

c. $$\angle H(\omega) = \begin{cases} -\frac{\omega}{2} + \frac{\pi}{2}, & \sin(\frac{\omega}{2}) \geq 0 \Leftrightarrow 4k\pi \leq \omega \leq 4k\pi + 2\pi, \quad k = 0, \pm 1, \cdots \\ -\frac{\omega}{2} + \frac{\pi}{2} + \pi, & \sin(\frac{\omega}{2}) < 0 \Leftrightarrow 4k\pi - 2\pi < \omega < 4k\pi, \quad k = 0, \pm 1, \pm 2, \cdots \end{cases}$$



2. (25 pts.) The signal $x[n] = \cos(2\pi n/3)$ is input to a digital filter with frequency response magnitude and phase as shown:





Find the output $y[n]$ from this system.

$$\because\ x[n] = \cos\left(\tfrac{2\pi n}{3}\right) = \tfrac{1}{2}\left(e^{j\frac{2\pi}{3}n} + e^{-j\frac{2\pi}{3}n}\right)$$ Complex exponential input

$$\therefore\ y[n] = H\left(\tfrac{2\pi}{3}\right)\frac{e^{j\frac{2\pi}{3}n}}{2} + H\left(-\tfrac{2\pi}{3}\right)\frac{e^{-j\frac{2\pi}{3}n}}{2}$$

From $|H(\omega)| \Rightarrow \left|H\left(\tfrac{2\pi}{3}\right)\right| = \tfrac{1}{3}\cdot 1 = \tfrac{1}{3}$

From $\angle H(\omega) \Rightarrow \angle H\left(\tfrac{2\pi}{3}\right) = \tfrac{2}{3}\times\left(-\tfrac{\pi}{2}\right) = -\tfrac{\pi}{3}$

$\Rightarrow H\left(\tfrac{2\pi}{3}\right) = \tfrac{1}{3}e^{-j\frac{\pi}{3}}$

Since $H(\omega) = H(\omega + 2\pi) \Rightarrow H\left(-\tfrac{2\pi}{3}\right) = H\left(2\pi - \tfrac{2\pi}{3}\right) = H\left(\tfrac{4\pi}{3}\right)$

Similarly, we can find $\left|H\left(\tfrac{4}{3}\pi\right)\right| = \tfrac{1}{3}$ $\angle H\left(\tfrac{4}{3}\pi\right) = \tfrac{\pi}{3} \Rightarrow H\left(\tfrac{4\pi}{3}\right) = \tfrac{1}{3}e^{j\frac{\pi}{3}}$

$$\therefore\ y[n] = \tfrac{1}{6}e^{-j\frac{\pi}{3}}\cdot e^{j\frac{2\pi}{3}n} + \tfrac{1}{6}e^{j\frac{\pi}{3}}\cdot e^{-j\frac{2\pi}{3}n}$$

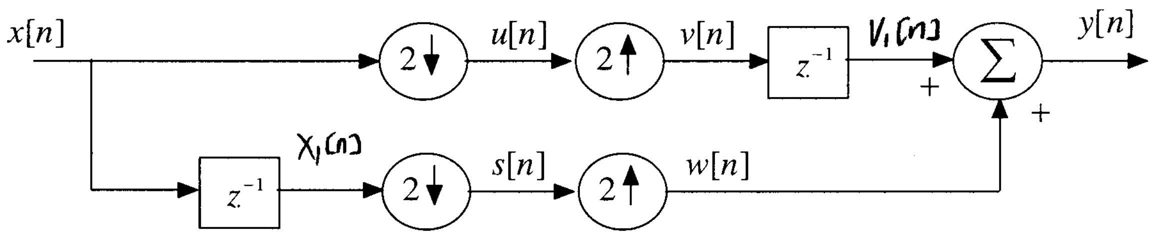
$$= \tfrac{1}{6}\left(e^{j\left(\frac{2\pi}{3}n - \frac{\pi}{3}\right)} + e^{-j\left(\frac{2\pi}{3}n - \frac{\pi}{3}\right)}\right) = \underline{\underline{\tfrac{1}{3}\cos\left(\tfrac{2\pi}{3}n - \tfrac{\pi}{3}\right)}}$$

Actually we can directly compute $y[n] = \left|H\left(\tfrac{2\pi}{3}\right)\right|\cdot\cos\left(\tfrac{2\pi}{3}n + \angle H\left(\tfrac{2\pi}{3}\right)\right)$

$$\Rightarrow y[n] = \underline{\underline{\tfrac{1}{3}\cos\left(\tfrac{2\pi}{3}n - \tfrac{\pi}{3}\right)}}$$

Same result.

3. (25) Consider the system below where z^{-1} denotes a unit sample delay.



with input signal

n	...,	−2,	−1,	0,	1,	2,	3,	4,	5,	6,	...,
$x[n]$	...,	0,	0,	5,	4,	3,	2,	1,	0,	0,	...,

Tabulate the signals $u[n], v[n], s[n], w[n], y[n]$.

n	...	−2	−1	0	1	2	3	4	5	6	...
$x[n]$	...	0	0	5	4	3	2	1	0	0	...
$u[n]$	...	0	0	5	3	1	0	0	0	0	...
$v[n]$	...	0	0	5	0	3	0	1	0	0	...
$\boxed{v_1[n]}$	...	0	0	0	5	0	3	0	1	0	...
$x_1[n]$	...	0	0	0	5	4	3	2	1	0	...
$s[n]$	...	0	0	0	4	2	0	0	0	0	...
$\boxed{w[n]}$	...	0	0	0	0	4	0	2	0	0	...
$y[n]$	...	0	0	0	5	4	3	2	1	0	...

Upper branch:

$u[n] = x[2n]$

$$v[n] = \begin{cases} u[\frac{n}{2}], & n=2k,\ k=0,\pm1,\pm2,\ldots \\ 0, & \text{o.w.} \end{cases} = \begin{cases} x[2\cdot\frac{n}{2}], & n=2k \\ 0 & \text{o.w} \end{cases} = \begin{cases} x[n], & n=2k \\ 0, & \text{o.w} \end{cases}$$

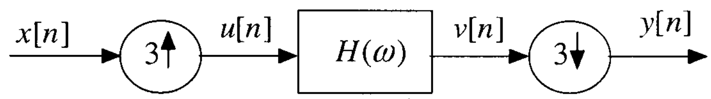
$$v_1[n] = v[n-1] = \begin{cases} x[n-1], & n-1=2k \\ 0, & \text{o.w.} \end{cases} = \begin{cases} x[n-1], & n=2k+1 \text{ (odd integers)} \\ 0, & \text{o.w.} \end{cases}$$

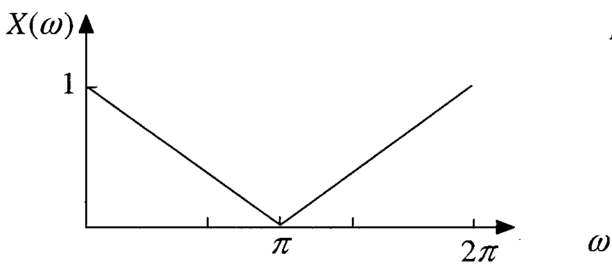
Lower branch: $x_1[n] = x[n-1]$

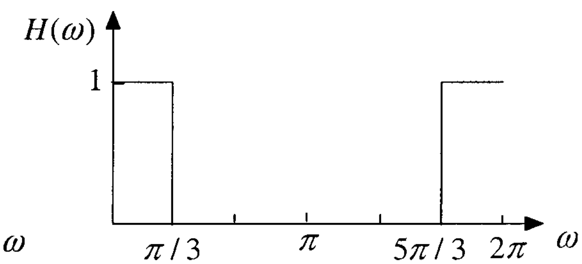
$$s[n] = x_1[2n] = x[2n-1], \quad w[n] = \begin{cases} s[\frac{n}{2}], & n=2k \\ 0, & \text{o.w} \end{cases} = \begin{cases} x[2\cdot\frac{n}{2}-1], & n=2k \\ 0, & \text{o.w} \end{cases} = \begin{cases} x[n-1], & n \text{ even} \\ 0, & \text{o.w.} \end{cases}$$

$\therefore\ y[n] = v_1[n] + w[n] = \underline{\underline{x[n-1]}}$ for any n

4. (25 pts) Consider the DT system and input signal $x[n]$ with DTFT $X(\omega)$ and filter frequency response $H(\omega)$ shown below:



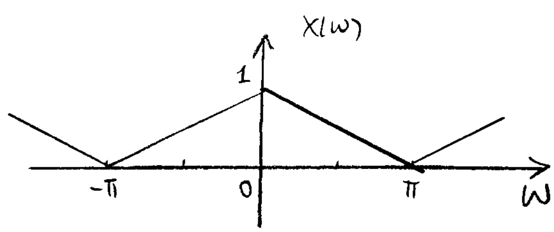


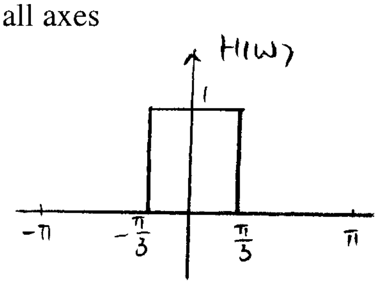


a) (8) Sketch the DTFT $U(\omega)$. Be sure to label all axes

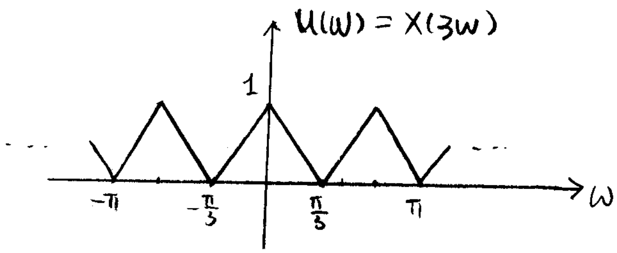
b) (8) Sketch the DTFT $V(\omega)$. Be sure to label all axes

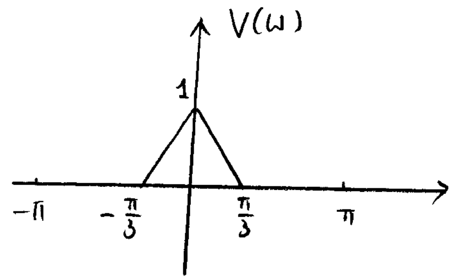
c) (9) Sketch the DTFT $Y(\omega)$. Be sure to label all axes

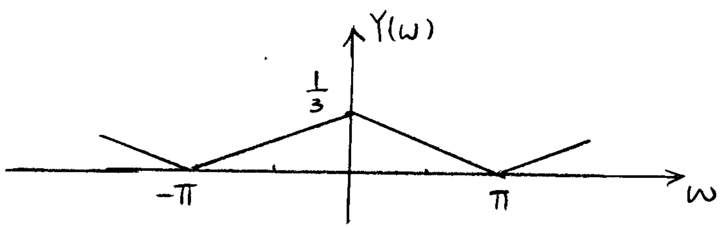




$X(\omega)$, $H(\omega)$ periodic with 2π draw $X(\omega)$, $H(\omega)$ for $\omega \in [-\pi, \pi]$







$$Y(\omega) = \frac{1}{3}\sum_{k=0}^{2} V\left(\frac{\omega - 2k\pi}{3}\right)$$