14 April 2000 **Name:** SOLUTION

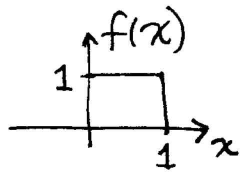
EE 438 **Exam No. 3** **Spring 2000**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Let X and Y be two independent random variables which are each uniformly distributed on the interval [0,1). Define a third random variable Z=XY.
   a. (12 pts.) Find the mean and variance of Z.
   b. (13 pts.) Find the correlation coefficient between Z and X.

a) $\bar{Z} = E[Z] = E[XY] = E[X]E[Y]$

because X & Y are independent



$E[X] = \frac{1}{2}$, by inspection ($\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$)

Similarly, $E[Y] = \frac{1}{2}$

$\Rightarrow \boxed{\bar{Z} = \frac{1}{4}}$

$\sigma_Z^2 = \mathrm{var}[Z] = E[Z^2] - (E[Z])^2 = E[X^2Y^2] - \frac{1}{16} = E[X^2]E[Y^2] - \frac{1}{16}$

$E[X^2] = E[Y^2] = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$\sigma_Z^2 = \frac{1}{9} - \frac{1}{16} = \frac{16-9}{144} = \frac{7}{144}$

$\Rightarrow \boxed{\sigma_Z^2 = \frac{7}{144}}$

b) $\rho_{ZX} = \frac{\sigma_{ZX}^2}{\sigma_Z \sigma_X} = \frac{E[ZX] - E[Z]E[X]}{\sigma_Z \sigma_X}$

1. (continued)

b) (cont.)

$$E[ZX] = E[X^2 Y] = E[X^2]\,E[Y] = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad\Rightarrow\quad \sigma_X = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$$

$$\sigma_Z^2 = \frac{7}{144} \quad\Rightarrow\quad \sigma_Z = \frac{\sqrt{7}}{12}$$

$$\rho_{ZX} = \frac{\frac{1}{6} - \frac{1}{4}\cdot\frac{1}{2}}{\frac{\sqrt{7}}{12}\cdot\frac{1}{2\sqrt{3}}} = \frac{\frac{4-3}{24}}{\frac{1}{24}\sqrt{\frac{7}{3}}} = \sqrt{\frac{3}{7}}$$

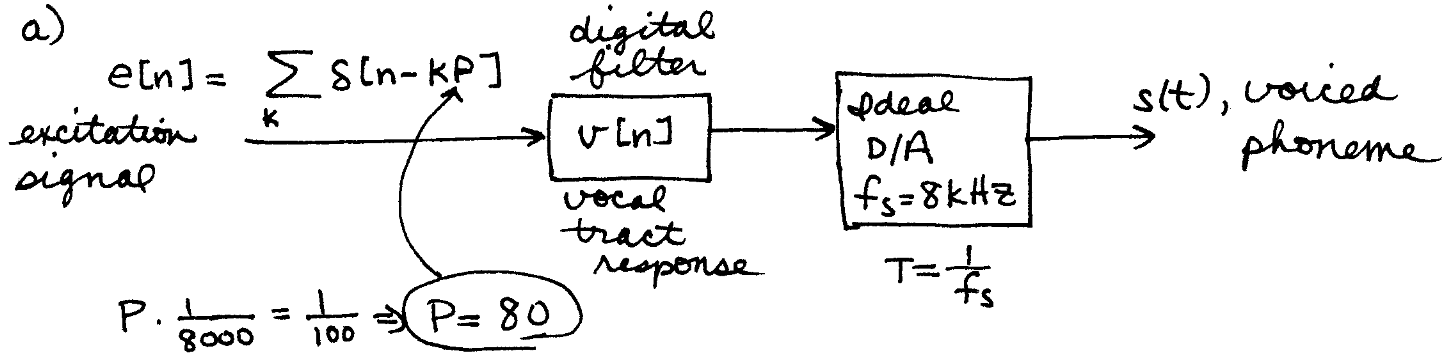
$$\Rightarrow \boxed{\rho_{ZX} = \sqrt{\frac{3}{7}}}$$

2. (20 pts.) Design a complete digital system that will generate a voiced phoneme with a pitch frequency of 100 Hz, a first formant frequency of 2 kHz and a weaker second formant frequency of 1 kHz. The digital system is to operate at a sampling frequency of 8 kHz. Be sure to show full details of the system, including:
   a. (10 pts.) a fully labeled block diagram
   b. (7 pts.) pole and zero locations for any digital filters that are required.

   As in any design problem, you will have to make some choices. I have not provided sufficient information to uniquely specify the solution.

   c. (3 pts.) For the signal generated by your system above, sketch what a narrowband spectrogram would look like.

a)



b) $f_1 = 2\,kHz$ $\qquad \omega_1 = 2\pi f_1 T = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$

$f_2 = 1\,kHz$ $\qquad \omega_2 = 2\pi f_2 T = 2\pi \frac{1}{8} = \frac{\pi}{4}$

The formant at ω_2 is weaker than that at ω_1, so its magnitude must be smaller. If a complex pole is present in the system, its complex conjugate must also be present.

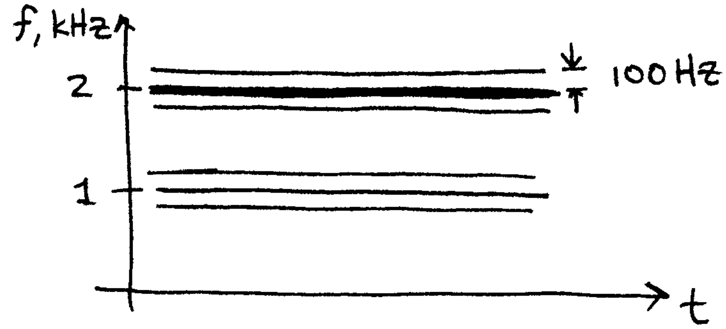
Zeroes: none necessary

poles: $0.9e^{j\frac{\pi}{2}}$, $0.9e^{-j\frac{\pi}{2}}$, $0.7e^{j\frac{\pi}{4}}$, $0.7e^{-j\frac{\pi}{4}}$

$$V(e^{j\omega}) = \frac{1}{(e^{j\omega} - 0.9e^{j\frac{\pi}{2}})(e^{j\omega} - 0.9e^{-j\frac{\pi}{2}})(e^{j\omega} - 0.7e^{j\frac{\pi}{4}})(e^{j\omega} - 0.7e^{-j\frac{\pi}{4}})}$$

2. (continued)

c)



3. (30 pts) Consider the STDTFT defined as

$$X(\omega, n) = \sum_k x[k]w[n-k]e^{-jwk},$$

where $x[n]$ is the speech signal and $w[n]$ is the window sequence. Let

$$x[n] = \begin{cases} \cos(\pi n/3), & n = -100,\ldots,100 \\ 0, & \text{else} \end{cases}$$

and

$$w[n] = \begin{cases} 1, & n = -10,\ldots,10 \\ 0, & \text{else} \end{cases}$$

a. (20 pts.) Find the magnitude of the STDTFT for $|n| < 90$.

b. (2 pts.) Find the magnitude of the STDTFT for $|n| > 110$.

c. (3 pts.) Sketch the spectrogram corresponding to $X(\omega, n)$ for all n.

Now suppose we define a new signal $y[n]$ according to

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-400k]$$

where $x[n]$ is defined as above.

d. (4 pts.) Sketch the spectrogram corresponding to this signal for all n.

e. (1 pts.) Is it a wideband or narrowband spectrogram?

a) For $|n| < 90$, all of the window is in an area of x which is non-zero. The magnitude is approximately the same. For ease, let $n=0$ and solve for the STDTFT. The result is real valued, and therefore the magnitude is simply its absolute value.

$$X(\omega, 0) = \sum_{k=-10}^{10} \cos\left(\frac{\pi k}{3}\right) e^{-j\omega k}$$

$$= \frac{1}{2} \sum_{k=-10}^{10} \left\{ e^{j\frac{\pi k}{3}} e^{-j\omega k} + e^{-j\frac{\pi}{3}k} e^{-j\omega k} \right\}$$

aside: $$\sum_{k=-N}^{N} e^{-j(\omega-\omega_0)k} = \sum_{l=0}^{2N} e^{-j(\omega-\omega_0)(l-N)}$$

$$= e^{+j(\omega-\omega_0)N} \cdot \frac{1-e^{-j(\omega-\omega_0)(2N+1)}}{1-e^{-j(\omega-\omega_0)}}$$

3. (continued)

a)(cont.)

$$\sum_{k=-N}^{N} e^{-j(\omega-\omega_0)k} = e^{j(\omega-\omega_0)N} \cdot \frac{e^{-j(\omega-\omega_0)\left(\frac{2N+1}{2}\right)} \sin\left((\omega-\omega_0)\frac{2N+1}{2}\right)}{e^{-j(\omega-\omega_0)\frac{1}{2}} \sin\left((\omega-\omega_0)\frac{1}{2}\right)}$$

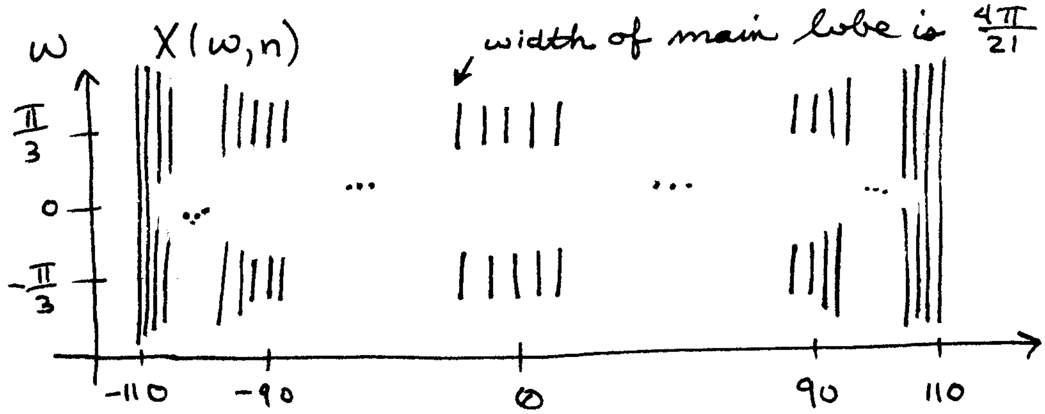
$$= \frac{\sin\left((\omega-\omega_0)\frac{2N+1}{2}\right)}{\sin\left((\omega-\omega_0)\frac{1}{2}\right)}$$

$$X(\omega, 0) = \frac{1}{2}\left\{ \frac{\sin\left((\omega-\frac{\pi}{3})\frac{21}{2}\right)}{\sin\left((\omega-\frac{\pi}{3})\frac{1}{2}\right)} + \frac{\sin\left((\omega+\frac{\pi}{3})\frac{21}{2}\right)}{\sin\left((\omega+\frac{\pi}{3})\frac{1}{2}\right)} \right\}$$

Magnitude of the STDTFT is the magnitude of $X(\omega, 0)$ given above.

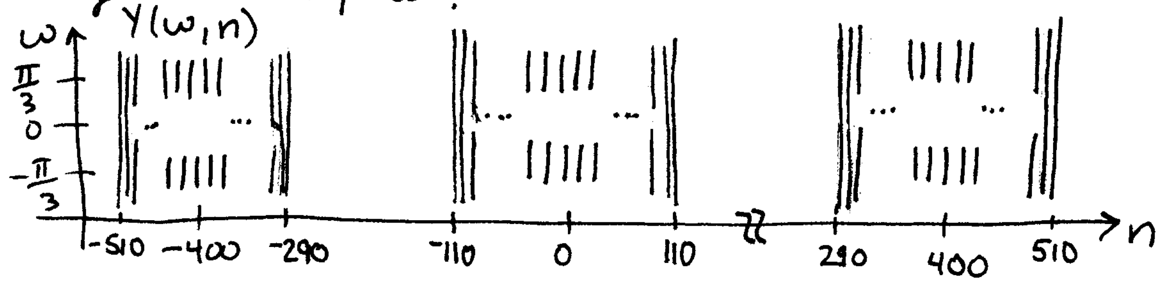
b) The magnitude of the STDTFT for $|n| > 110$ is zero.

c)



As the spectrogram goes from 90 to 110, the main lobe width increases and the height decreases.

d) The sketch is the same as in c), except it repeats every 400 samples.



e) wideband

4. (25 pts.) In class, we talked about linear prediction in a deterministic setting. It is also possible to define a linear predictor for a random signal. Let $X[n]$ be a wide-sense stationary random signal with the following autocorrelation:

$$r_{XX}[n] \equiv E\{X[m]X[m+n]\} = \begin{cases} 1, & n=0, \\ 2/3, & |n|=1, \\ 1/3, & |n|=2, \\ 0, & \text{else}. \end{cases}$$

Let $\hat{X}[n]$ be a linear predictor for $X[n]$, based on the observed values of $X[n-1]$ and $X[n-2]$, i.e.

$$\hat{X}[n] = aX[n-1] + bX[n-2],$$

where a and b are constants. Find values for a and b that will minimize the expected mean-squared error:

$$\varepsilon = E\left\{\left|\hat{X}[n] - X[n]\right|^2\right\}.$$

$$\varepsilon = E\left\{(aX[n-1] + bX[n-2] - X[n])^2\right\}$$

$$= E\left\{a^2X^2[n-1] + 2abX[n-1]X[n-2] - 2aX[n-1]X[n] + b^2X^2[n-2] - 2bX[n-2]X[n] + X^2[n]\right\}$$

$$= a^2 + 2ab\cdot\tfrac{2}{3} - 2a\cdot\tfrac{2}{3} + b^2 - 2b\cdot\tfrac{1}{3} + 1$$

$$\frac{\partial \varepsilon}{\partial a} = 2a + \tfrac{4}{3}b - \tfrac{4}{3} = 0 \Rightarrow 2a + \tfrac{4}{3}b = \tfrac{4}{3}$$

$$\frac{\partial \varepsilon}{\partial b} = \tfrac{4}{3}a + 2b - \tfrac{2}{3} = 0 \Rightarrow \tfrac{4}{3}a + 2b = \tfrac{2}{3}$$

$$\begin{bmatrix} 2 & \frac{4}{3} \\ \frac{4}{3} & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \left(\begin{bmatrix} 2 & \frac{4}{3} \\ \frac{4}{3} & 2 \end{bmatrix}\right)^{-1}\begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & \frac{4}{3} \\ \frac{4}{3} & 2 \end{bmatrix}\right)^{-1} = \frac{1}{4 - \frac{16}{9}}\begin{bmatrix} 2 & -\frac{4}{3} \\ -\frac{4}{3} & 2 \end{bmatrix} = \frac{9}{20}\begin{bmatrix} 2 & -\frac{4}{3} \\ -\frac{4}{3} & 2 \end{bmatrix} = \begin{bmatrix} \frac{9}{10} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{9}{10} \end{bmatrix}$$

$$\frac{36-16}{9} = \frac{20}{9}$$

4. (continued)

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{9}{10} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{9}{10} \end{bmatrix} \begin{bmatrix} 4/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} - \frac{2}{5} \\ -\frac{4}{5} + \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{1}{5} \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{array}{l} a = \frac{4}{5} \\ b = -\frac{1}{5} \end{array}}$$