10 March 2000 **Name:** SOLUTION

EE 438 **Exam No. 2** **Spring 2000**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider a DT linear, time-invariant system with the following transfer function:

$$H(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)},\quad \frac{1}{4}<|z|<\frac{3}{4}$$

   a. (10 pts.) Use the graphical approach to compute the magnitude of the frequency response at $\omega=\pi/2$.

   b. (15 pts.) Find the impulse response $h[n]$.

a) $$H(z)=\frac{z^2}{(z-\frac{1}{4})(z+\frac{3}{4})}$$

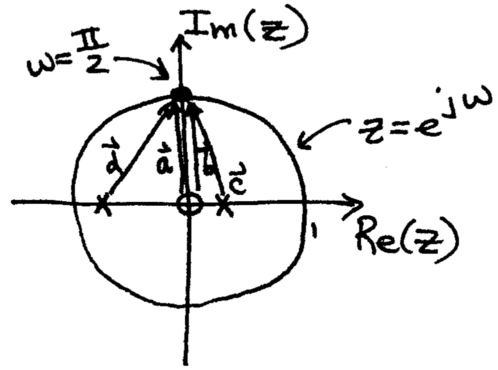
zeros: $0, 0$
poles: $\frac{1}{4}, -\frac{3}{4}$

$$H(e^{j\omega})=\frac{e^{j\omega}\cdot e^{j\omega}}{(e^{j\omega}-\frac{1}{4})(e^{j\omega}+\frac{3}{4})}=\frac{\vec{a}\cdot\vec{b}}{\vec{c}\cdot\vec{d}}$$

$$|H(e^{j\omega})|=\frac{|\vec{a}|\cdot|\vec{b}|}{|\vec{c}|\cdot|\vec{d}|}$$

$$\angle H(e^{j\omega})=\angle\vec{a}+\angle\vec{b}-\angle\vec{c}-\angle\vec{d}$$ ← not required

at $\omega=\frac{\pi}{2}$:



$\vec{a}=j \qquad |\vec{a}|=1$

$\vec{b}=j \qquad |\vec{b}|=1$

$\vec{c}=-\frac{1}{4}+j \qquad |\vec{c}|=\sqrt{\frac{1}{16}+1}=\frac{\sqrt{17}}{4}$

$\vec{d}=\frac{3}{4}+j \qquad |\vec{d}|=\sqrt{\frac{9}{16}+1}=\frac{5}{4}$

1. (continued)

a) cont.

$$\boxed{|H(e^{j\frac{\pi}{2}})| = \frac{1 \cdot 1}{\frac{\sqrt{17}}{4} \cdot \frac{5}{4}} = \frac{16}{5\sqrt{17}} \approx 0.776}$$

b) $$H(z) = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1+\frac{3}{4}z^{-1}}$$

$$A = \left.\frac{1}{1+\frac{3}{4}z^{-1}}\right|_{z^{-1}=4} = \frac{1}{1+3} = \frac{1}{4}$$

$$B = \left.\frac{1}{1-\frac{1}{4}z^{-1}}\right|_{z^{-1}=-\frac{4}{3}} = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$$

Since $\frac{1}{4} < |z| < \frac{3}{4}$, use the causal inverse z-transform for $\frac{A}{1-\frac{1}{4}z^{-1}}$ and the anti-causal inverse ZT for $\frac{B}{1+\frac{3}{4}z^{-1}}$.

$$\boxed{h[n] = \frac{1}{4}\left(\frac{1}{4}\right)^n u[n] - \frac{3}{4}\left(-\frac{3}{4}\right)^n u[-n-1]}$$

Note that $|h[n]|$ goes to ∞ as $n \to -\infty$. This instability is expected because the unit circle is not contained in the ROC.

2. (25 pts.) Consider an N point signal $x[n], n=0,\ldots,N-1$ with N point DFT $X[k], k=0,\ldots,N-1$. Find the N point DFT $Y[k], k=0,\ldots,N-1$ in terms of $X[k], k=0,\ldots,N-1$, when

   a. (10 pts.) $y[n]=x[n]\cos(\pi n/4)$,

   b. (15 pts.) $y[n]=x[N-1-n], n=0,\ldots,N-1$.

a) $$y[n] = x[n]\cdot\frac{1}{2}\left(e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}\right)$$

$$= \frac{1}{2}\left(e^{\frac{j2\pi n\left(\frac{N}{8}\right)}{N}} x[n] + e^{-j\frac{2\pi n\left(\frac{N}{8}\right)}{N}} x[n]\right)$$

Use the modulation rule and assume $\frac{N}{8}$ is an integer.

$$Y[k] = \frac{1}{2}\left(X[k-\tfrac{N}{8}] + X[k+\tfrac{N}{8}]\right)$$

Since, ideally, the argument of $X[\cdot]$ should be in the range $0,\ldots,N-1$,

$$Y[k] = \frac{1}{2}\left(X[k-\tfrac{N}{8}] + X[k-(N-\tfrac{N}{8})]\right)$$

2. (25 pts.) Consider an N point signal $x[n], n=0,\dots,N-1$ with N point DFT $X[k], k=0,\dots,N-1$. Find the N point DFT $Y[k], k=0,\dots,N-1$ in terms of $X[k], k=0,\dots,N-1$, when

   a. (10 pts.) $y[n]=x[n]\cos(\pi n/4)$,

   b. (15 pts.) $y[n]=x[N-1-n], n=0,\dots,N-1$.

alternate solution: (without assuming $\frac{N}{8}$ is an integer)

a) The product rule for DFT gives

$$Y[k] = X[k] \circledast \text{DFT}\{\cos(\tfrac{\pi n}{4})\}$$

Using the direct approach:

$$Y[k] = \sum_{n=0}^{N-1} x[n]\cos(\tfrac{\pi n}{4})\, e^{-j\frac{2\pi k n}{N}}$$

Use $x[n] = \frac{1}{N}\sum_{\ell=0}^{N-1} X[\ell]\, e^{j\frac{2\pi \ell n}{N}}$ and $\cos(\frac{\pi n}{4}) = \frac{1}{2}\{e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}\}$

$$Y[k] = \sum_{n=0}^{N-1}\frac{1}{N}\sum_{\ell=0}^{N-1} X[\ell]\, e^{j\frac{2\pi \ell n}{N}}\,\frac{1}{2}\{e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}\}\, e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{2N}\sum_{\ell=0}^{N-1} X[\ell]\sum_{n=0}^{N-1}\left\{e^{j\frac{2\pi}{N}(\frac{N}{8}+\ell-k)n} + e^{j\frac{2\pi}{N}(-\frac{N}{8}+\ell-k)n}\right\}$$

$$= \frac{1}{2N}\sum_{\ell=0}^{N-1} X[\ell]\left\{\frac{1-e^{j2\pi(\frac{N}{8}+\ell-k)}}{1-e^{j\frac{2\pi}{N}(\frac{N}{8}+\ell-k)}} + \frac{1-e^{j2\pi(-\frac{N}{8}+\ell-k)}}{1-e^{j\frac{2\pi}{N}(-\frac{N}{8}+\ell-k)}}\right\}$$

$$= \frac{1}{2N}\sum_{\ell=0}^{N-1} X[\ell]\left\{\frac{e^{j\frac{\pi N}{8}}\sin(\frac{\pi N}{8})}{e^{j\frac{\pi}{N}(\frac{N}{8}+\ell-k)}\sin(\frac{\pi}{N}(\frac{N}{8}+\ell-k))} + \frac{e^{-j\frac{\pi N}{8}}\sin(-\frac{\pi N}{8})}{e^{j\frac{\pi}{N}(-\frac{N}{8}+\ell-k)}\sin(\frac{\pi}{N}(-\frac{N}{8}+\ell-k))}\right\}$$

$$\Rightarrow Y[k] = \frac{e^{j\frac{\pi k}{N}}\sin(\frac{\pi N}{8})}{2N}\sum_{\ell=0}^{N-1} X[\ell]\left\{\frac{e^{j\pi(\frac{N-1}{8}-\ell)}}{\sin(\frac{\pi}{N}(\frac{N}{8}+\ell-k))} + \frac{e^{j\pi(\frac{1-N}{8}-\frac{\ell}{N})}}{\sin(\frac{\pi}{N}(-\frac{N}{8}+\ell-k))}\right\}$$

2. (continued)

b) $y[n] = x[N-1-n] \quad , n = 0, \ldots, N-1$

$$Y[k] = \sum_{n=0}^{N-1} x[N-1-n] e^{-j\frac{2\pi kn}{N}}$$

Let $\ell = N-1-n$

$$= \sum_{\ell=N-1}^{0} x[\ell] e^{-j\frac{2\pi k(N-1-\ell)}{N}}$$

$$= e^{-j\frac{2\pi k(N-1)}{N}} \sum_{\ell=0}^{N-1} x[\ell] e^{-j\frac{2\pi(-k)\ell}{N}}$$

$$\Rightarrow Y[k] = e^{-j\frac{2\pi k(N-1)}{N}} X[-k]$$

This is OK, but ideally we want the argument of $X[\cdot]$ to be in the range $0, \ldots, N-1$. Since $X[k]$ is periodic with period N,

$$Y[k] = \begin{cases} e^{-j\frac{2\pi k(N-1)}{N}} X[k] & , k=0 \\ e^{-j\frac{2\pi k(N-1)}{N}} X[N-k] , & k = 1, \ldots, N-1 \end{cases}$$

3. (25 pts) Consider the 16 point signal $x[n]=\cos(\pi n/8)$, $n=0,\ldots,15$.

   a. (3 pts.) Sketch $x[n]$.

   b. (5 pts.) Find the DFT $X[k]$ for $x[n]$.

   c. (2 pts.) Sketch $X[k]$.

   Now consider a new 16 point signal $y[n]=\cos(\pi n/16)$ $n=0,\ldots,15$.

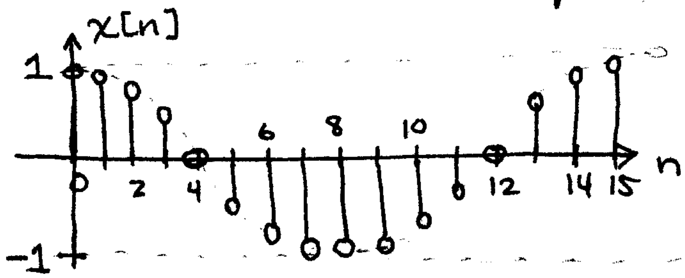
   d. (3 pts.) Sketch $y[n]$.

   e. (8 pts.) Find the DFT $Y[k]$ for $y[n]$.

   f. (4 pts.) Sketch $Y[k]$.

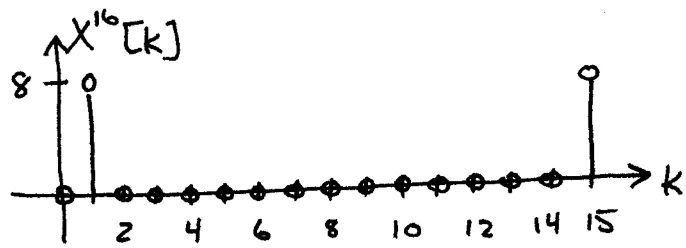
a) $x[n]=\cos\left(\frac{2\pi n}{16}\right)$, $n=0,\ldots,15$

x[n] traverses 1 cosine cycle in 16 samples.



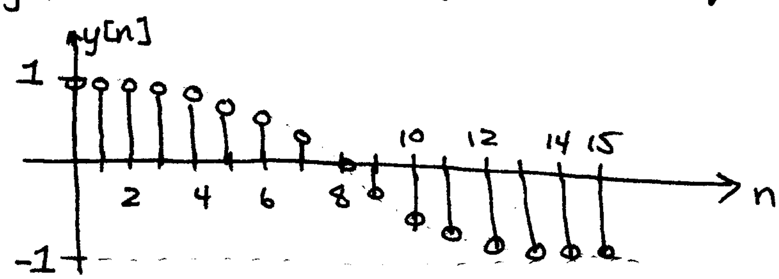
b) $X^{16}[k]=8\{\delta[k-1]+\delta[k-15]\}$ for $0\le k\le N-1$

c)



d) $y[n]=\cos\left(\frac{2\pi n(\frac{1}{2})}{16}\right)$, $n=0,\ldots,15$

y[n] traverses half a cosine cycle in 16 samples.



3. (continued)

e) $$Y^{(16)}[k] = \frac{1}{2}\sum_{n=0}^{15}\left(e^{j\frac{2\pi n(\frac{1}{2})}{16}} + e^{-j\frac{2\pi n(\frac{1}{2})}{16}}\right)e^{-j\frac{2\pi kn}{16}}$$

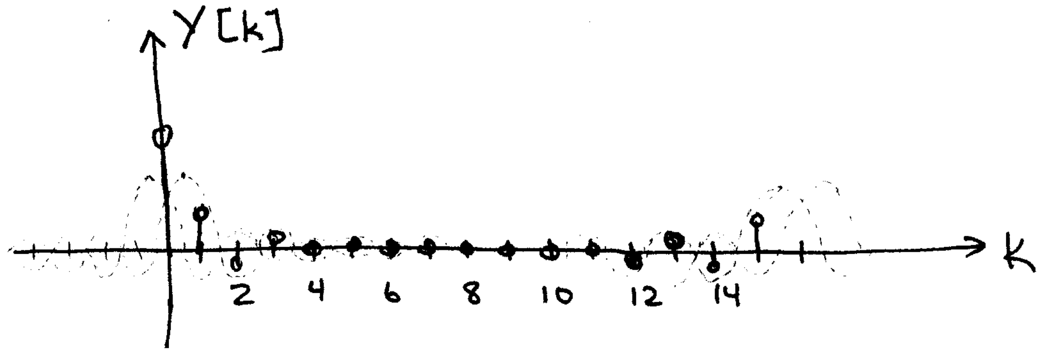
$$= \frac{1}{2}\sum_{n=0}^{15}\left(e^{-j\frac{2\pi(k-\frac{1}{2})n}{16}} + e^{-j\frac{2\pi(k+\frac{1}{2})n}{16}}\right)$$

$$= \frac{1}{2}\left\{\frac{1-e^{-j2\pi(k-\frac{1}{2})}}{1-e^{-j\frac{2\pi(k-\frac{1}{2})}{16}}} + \frac{1-e^{-j2\pi(k+\frac{1}{2})}}{1-e^{-j\frac{2\pi(k+\frac{1}{2})n}{16}}}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{-j\pi(k-\frac{1}{2})}\sin(\pi(k-\frac{1}{2}))}{e^{-j\frac{\pi(k-\frac{1}{2})}{16}}\sin(\frac{\pi}{16}(k-\frac{1}{2}))} + \frac{e^{-j\pi(k+\frac{1}{2})}\sin(\pi(k+\frac{1}{2}))}{e^{-j\frac{\pi(k+\frac{1}{2})}{16}}\sin(\frac{\pi}{16}(k+\frac{1}{2}))}\right\}$$

$$\Rightarrow Y^{(16)}[k] = \frac{1}{2}\left\{e^{-j\pi(k-\frac{1}{2})\frac{15}{16}}\frac{\sin(\pi(k-\frac{1}{2}))}{\sin(\frac{\pi}{16}(k-\frac{1}{2}))} + e^{-j\pi(k+\frac{1}{2})\frac{15}{16}}\frac{\sin(\pi(k+\frac{1}{2}))}{\sin(\frac{\pi}{16}(k+\frac{1}{2}))}\right\}$$

f)



4. (25 pts.)

   a. (15 pts.) Derive the equations for a 6 point fast Fourier transform (FFT).

   b. (10 pts.) Draw a complete flow diagram for your 6 point FFT being sure to label all quantities.

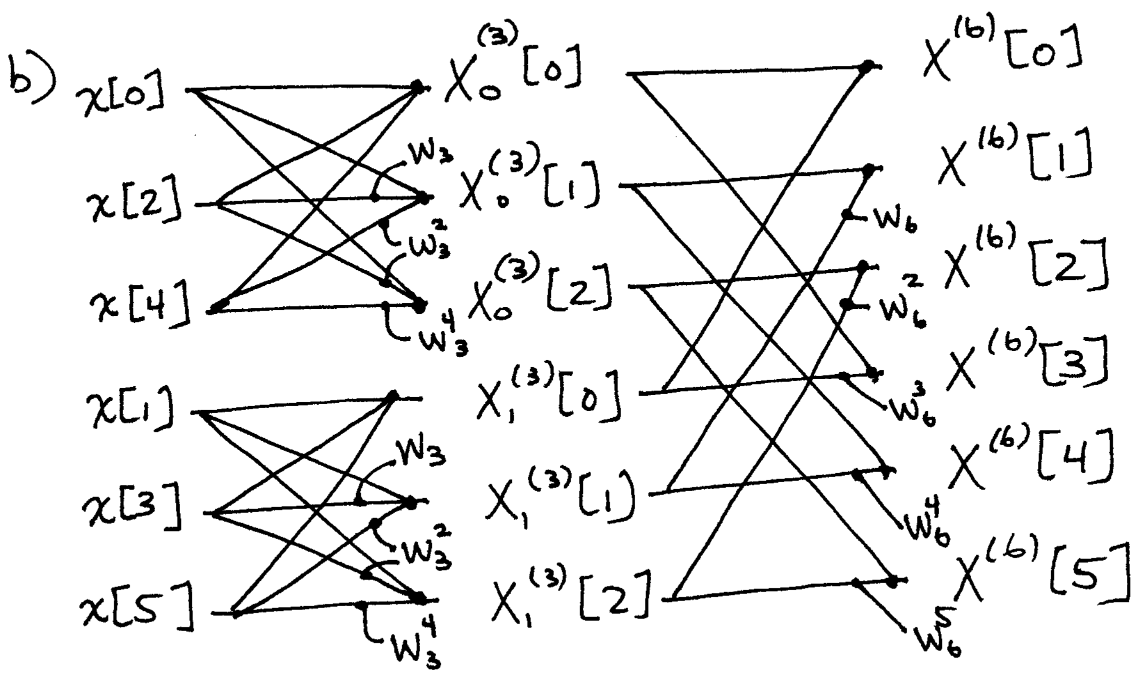
$$X^{(6)}[k] = \sum_{n=0}^{5} x[n] e^{-j\frac{2\pi kn}{6}}$$

Solution I:

a) Break into 2 3-pt. DFTs.

$$X^{(6)}[k] = \underbrace{\sum_{n=0}^{2} x[2n] e^{-j\frac{2\pi kn}{3}}}_{X_0^{(3)}[k]} + e^{-j\frac{2\pi k}{6}} \underbrace{\sum_{n=0}^{2} x[2n+1] e^{-j\frac{2\pi kn}{3}}}_{X_1^{(3)}[k]}$$

b)



$W_3 = e^{-j\frac{2\pi}{3}}$ $W_6 = e^{-j\frac{2\pi}{6}}$

4. (continued)

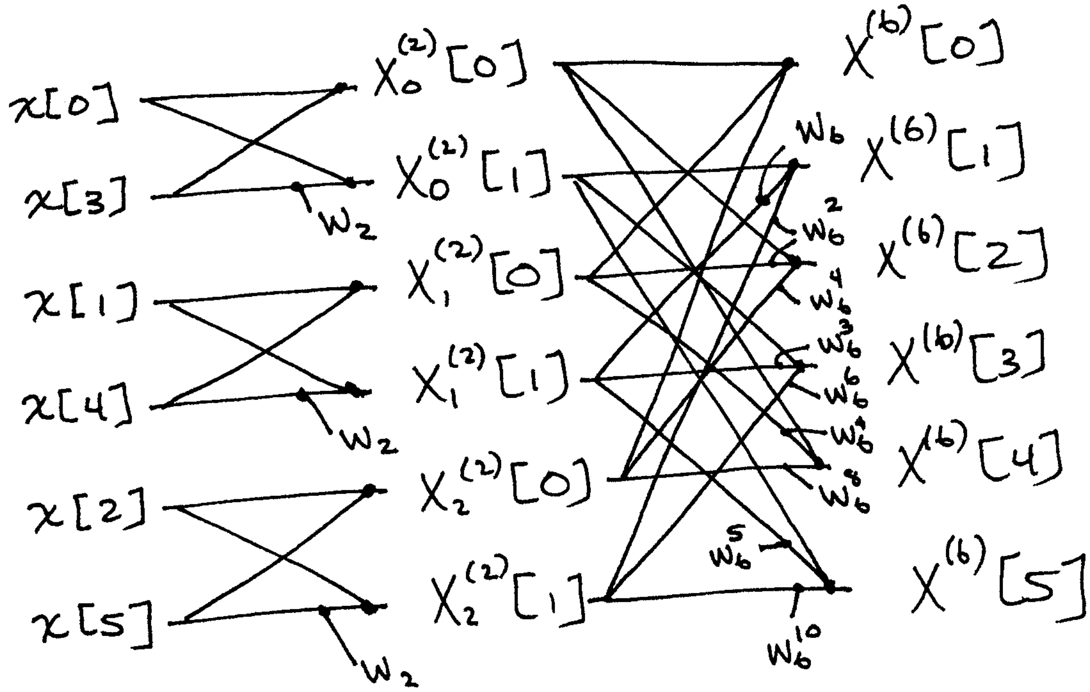
Solution II:

a) Break into 3 2-pt. DFTs:

$$X^{(6)}[k] = \underbrace{\sum_{n=0}^{1} x[3n] e^{-j\frac{2\pi kn}{2}}}_{X_0^{(2)}[k]} + e^{-j\frac{2\pi k}{6}} \underbrace{\sum_{n=0}^{1} x[3n+1] e^{-j\frac{2\pi kn}{n}}}_{X_1^{(2)}[k]}$$

$$+ e^{-j\frac{2\pi k2}{6}} \underbrace{\sum_{n=0}^{1} x[3n+2] e^{-j\frac{2\pi kn}{2}}}_{X_2^{(2)}[k]}$$

b)



$W_2 = e^{-j\frac{2\pi}{2}}$ $W_6 = e^{-j\frac{2\pi}{6}}$

1.	______
2.	______
3.	______
4.	______
Total	______