11 February 2000 **Name:** SOLUTION

EE 438 **Exam No. 1** **Spring 2000**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the system shown below with input $x[n]$ and output $y[n]$.

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$$

Suppose that this system is linear and time-invariant with impulse response $h[n] = \delta[n]$.

(5) a. Find an expression for the DTFT $Y(\omega)$ of the output in terms of the DTFT $X(\omega)$ of the input.

Now consider a system described by the relationship $y[n] = (-1)^n x[n]$.

(2) b. Is this system linear? (State "yes" or "no"; no proof is needed, and no partial credit will be given.)

(2) c. Is it time-invariant? (State "yes" or "no"; no proof is needed, and no partial credit will be given.)

(4) d. Find the impulse response for this system.

(12) e. Find an expression for the DTFT $Y(\omega)$ of the output in terms of the DTFT $X(\omega)$ of the input.

(a) $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 \Rightarrow \boxed{Y(e^{j\omega}) = X(e^{j\omega})}$

(b) Yes

(c) No

(d) $h[n] = (-1)^n \delta[n] = \delta[n] \Rightarrow \boxed{h[n] = \delta[n]}$

(e) $Y(e^{j\omega}) = \text{DTFT}\{(-1)^n x[n]\} = \text{DTFT}\{e^{j\pi n} x[n]\}$

$= X(e^{j(\omega - \pi)})$

$\Rightarrow \boxed{Y(e^{j\omega}) = X(e^{j(\omega - \pi)})}$

2. (25 pts.) Evaluate the convolution of the following pairs of signals:

(8) a. $rect(t)$ and $\delta(2t-1)$

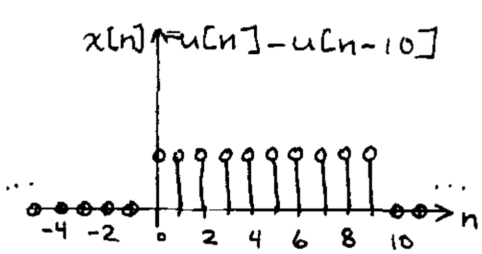
(17) b. $u[n]-u[n-10]$ and $2^{-n}u[n+10]$

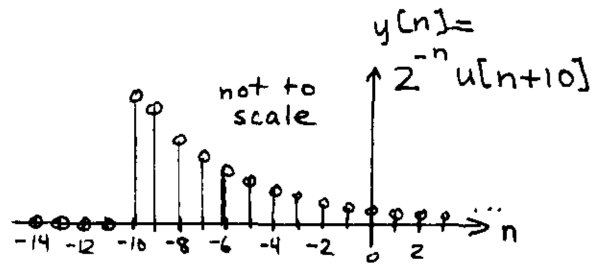
(a) $\int rect(t-\tau)\,\delta(2\tau-1)\,d\tau$

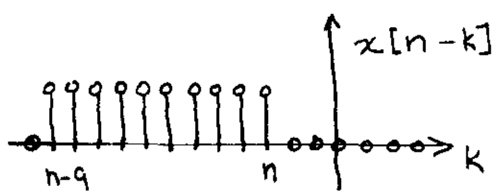
Let $\alpha = 2\tau$. $d\alpha = 2\,d\tau$

$$= \frac{1}{2}\int rect\left(t-\frac{\alpha}{2}\right)\delta(\alpha-1)\,d\alpha = \boxed{\frac{1}{2}\,rect\left(t-\frac{1}{2}\right)}$$

(b)







$$x[n] * y[n] = \sum_k x[n-k]\,y[k]$$

$n < -10$: $= 0$

$$-10 \le n \le -1 \; : \; = \sum_{k=-10}^{n} 2^{-k} = \sum_{\ell=0}^{n+10} 2^{-\ell+10} = 2^{10}\sum_{\ell=0}^{n+10}(2^{-1})^{\ell}$$

$\uparrow \ell = k+10$

2. (continued)

(b) $-10 \le n \le -1$: $= 2^{10} \dfrac{1-2^{-(n+11)}}{1-2^{-1}} = 2^{11}(1-2^{-n}2^{-11}) = 2^{11} - 2^{-n}$

$0 \le n$: $= \sum_{k=n-9}^{n} 2^{-k} = \sum_{\ell=0}^{9} 2^{-(\ell-9+n)} = 2^{(9-n)} \sum_{\ell=0}^{9} (2^{-1})^{\ell}$

$\ell = k + 9 - n$

$= 2^{(9-n)} \dfrac{1-2^{-10}}{1-2^{-1}} = 2^{(10-n)}(1-2^{-10})$

$= 2^{-n}(2^{10}-1)$

$$\Rightarrow \boxed{x[n] * y[n] = \begin{cases} 0, & n < -10 \\ 2^{11} - 2^{-n}, & -10 \le n \le -1 \\ 2^{-n}(2^{10}-1), & 0 \le n \end{cases}}$$

3 (25 pts.)The signal $x(t) = \cos(2\pi(5{,}000)t) + 0.5\cos(2\pi(10{,}000)t)$ is sampled with an ideal sampler at a rate of 15 kHz. The resulting digital signal $x[n]$ is then processed with the following DT system:

$$y[n] = 0.5(x[n] + x[n-2]).$$

(13) a. Find and sketch the DTFT $X(\omega)$ of the input to the system.

(12) b. Find and sketch the DTFT $Y(\omega)$ of the output of the system.

(a) $x[n] = \cos\left(\frac{2\pi}{3}n\right) + 0.5\cos\left(\frac{4\pi}{3}n\right)$

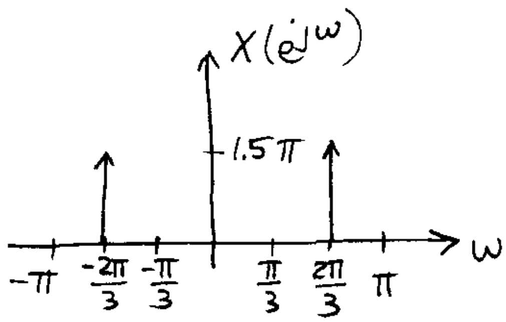
$$X(e^{j\omega}) = \pi \sum_k \Big\{ \delta\left(\omega - \tfrac{2\pi}{3} - 2\pi k\right) + \delta\left(\omega + \tfrac{2\pi}{3} - 2\pi k\right) + 0.5\left[\delta\left(\omega - \tfrac{4\pi}{3} - 2\pi k\right) + \delta\left(\omega + \tfrac{4\pi}{3} - 2\pi k\right)\right] \Big\}$$

$k=-1$ $k=1$ ← Aliasing!

For $|\omega| < \pi$,

$$X(e^{j\omega}) = \pi\Big\{\delta\left(\omega - \tfrac{2\pi}{3}\right) + \delta\left(\omega + \tfrac{2\pi}{3}\right) + 0.5\left[\delta\left(\omega + \tfrac{2\pi}{3}\right) + \delta\left(\omega - \tfrac{2\pi}{3}\right)\right]\Big\}$$

$$\boxed{X(e^{j\omega}) = \pi\Big\{1.5\,\delta\left(\omega - \tfrac{2\pi}{3}\right) + 1.5\,\delta\left(\omega + \tfrac{2\pi}{3}\right)\Big\}, \text{ and repeat every } 2\pi}$$



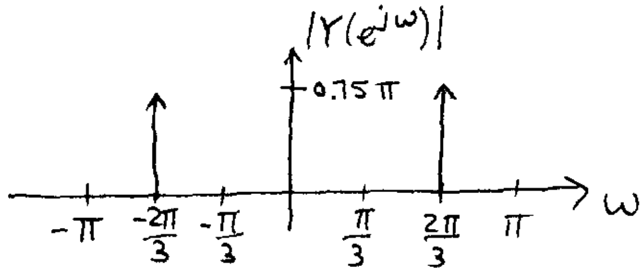
3. (continued)

(b) $Y(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + e^{-j\omega 2} X(e^{j\omega})\right)$

$= \frac{1}{2} e^{-j\omega}\left(e^{j\omega} + e^{-j\omega}\right) X(e^{j\omega})$

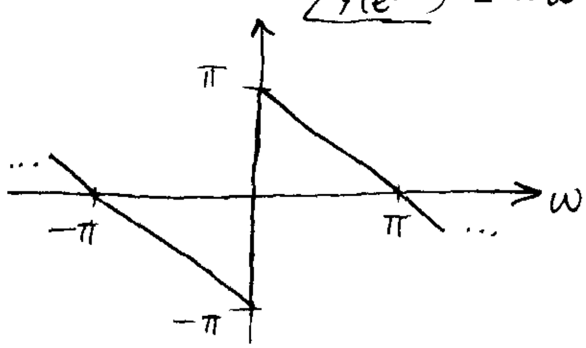
$= e^{-j\omega} \cos(\omega)\, X(e^{j\omega})$

$\Rightarrow Y(e^{j\omega}) = 1.5 \cos\left(\frac{2\pi}{3}\right) e^{-j\omega}\, \pi \left\{ \delta\left(\omega - \frac{2\pi}{3}\right) + \delta\left(\omega + \frac{2\pi}{3}\right) \right\}$, repeat every 2π

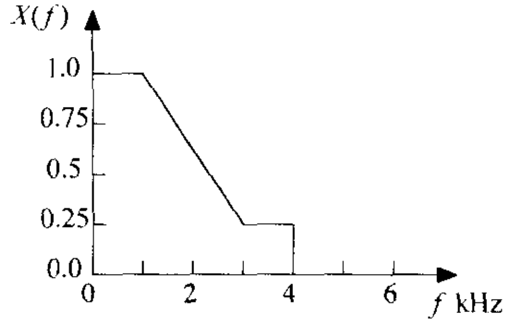


$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$\angle Y(e^{j\omega}) = -\omega \pm \pi$



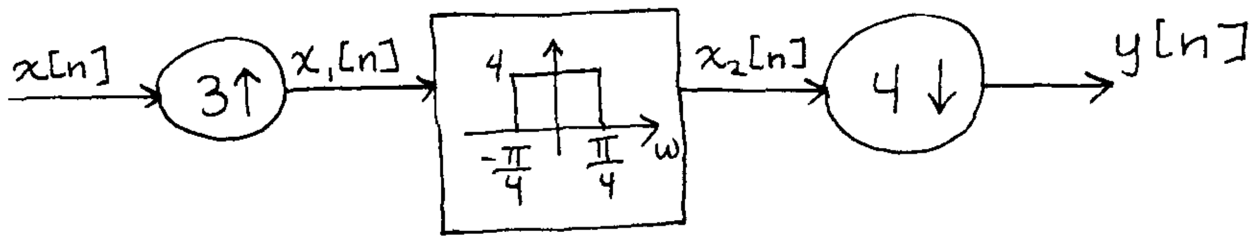
4. (25 pts.) Consider the CT signal $x(t)$ with CTFT $X(f)$ shown below:



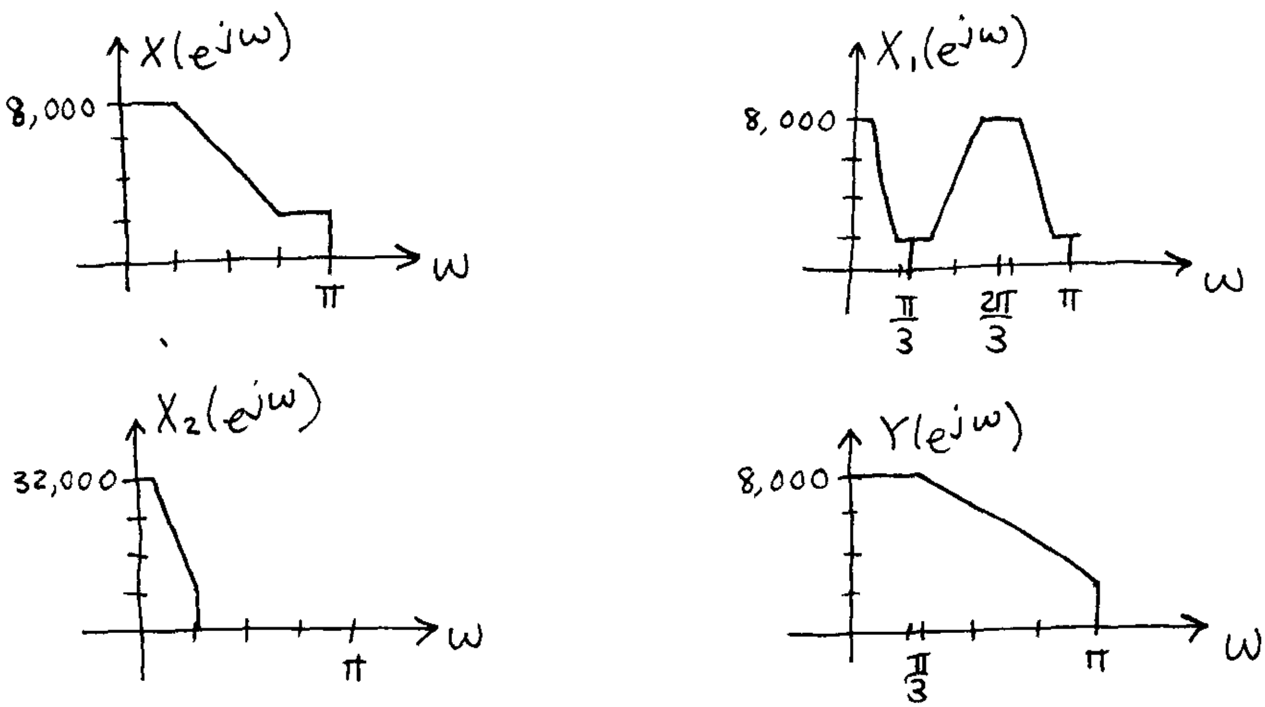
Suppose we sample this signal at the Nyquist rate of 8 kHz to generate the DT signal $x[n]$.
Design a DT system to lower the sampling rate to 75% of its initial value. The output of this system will be a signal $y[n]$ that is effectively sampled at 6 kHz. Your system should be designed so that no aliasing (folding over) of frequencies occurs. Some truncation of higher frequencies may be necessary.

a. Sketch a complete block diagram of your system showing the necessary parameters for all components of the system.
b. Sketch the DTFT spectrum of the signal at each point in your system.

(a)



(b)



4. (continued)

(b) Also acceptable (assuming $X(f)$ is one-sided):