December 13, 1999 **Name:** Solution

**EE 438** **Final Exam** **Fall 1999**

- You have 120 minutes to work the following five problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the system described by the following difference equation:

$$y[n] = \tfrac{1}{4}\{x[n] + 2x[n-1] + x[n-2]\}.$$

a. (17) Find simple expressions for the magnitude $|H(e^{j\omega})|$ of the frequency response and the phase $\angle H(e^{j\omega})$ of the frequency response.

b (8) Sketch $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$.

a) $$Y(e^{j\omega}) = \tfrac{1}{4}\{X(e^{j\omega}) + 2X(e^{j\omega})e^{-j\omega} + X(e^{j\omega})e^{-j2\omega}\}$$

$$= \tfrac{1}{4}\{1 + 2e^{-j\omega} + e^{-j2\omega}\}X(e^{j\omega})$$

$$H(e^{j\omega}) = \tfrac{1}{4}\{1 + 2e^{-j\omega} + e^{-j2\omega}\}$$

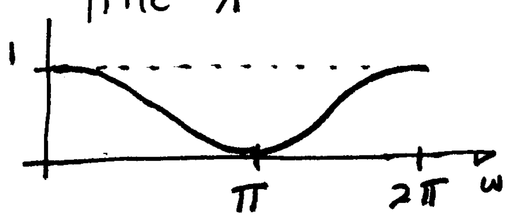
$$= \tfrac{1}{4}e^{-j\omega}\{e^{j\omega} + 2 + e^{-j\omega}\}$$

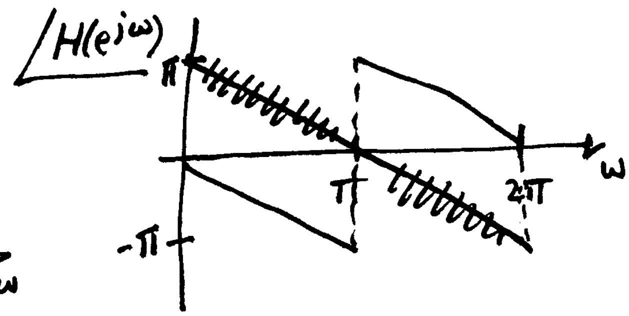
$$= \tfrac{1}{4}e^{-j\omega}\{2 + 2\cos(\omega)\}$$

$$|H(e^{j\omega})| = \tfrac{1}{2} + \tfrac{1}{2}\cos(\omega)$$

$$\angle H(e^{j\omega}) = -\omega, \quad \text{since } 2 + 2\cos(\omega) \geq 0 \ \forall\, \omega$$

b) $|H(e^{j\omega})|$



2. (25 pts.) Consider the system described by the following difference equation:

$$y[n] = x[n] - x[n-1] + \tfrac{5}{2}y[n-1] - y[n-2].$$

a. (6) By direct substitution into the difference equation, find the response $y[n],\ 0 \le n \le 5$ to the input

$$x[n] = \begin{cases} 1, & n = 0 \\ -1, & n = 1 \\ 0, & \text{else} \end{cases}$$

Assume that the system is initially at rest.

b. (7) Plot the locations of the poles and zeros for this filter.

c. (2) Is the system stable?

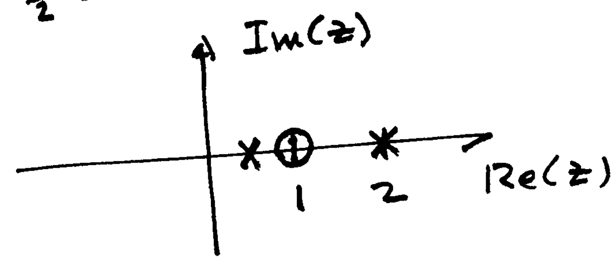
d. (10) Find a simple closed form expression for the impulse response of this system.

a)

| $n$ | $x[n]$ | $x[n-1]$ | $y[n-1]$ | $y[n-2]$ | $y[n]$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | -1 | 1 | 1 | 0 | $\frac{1}{2}$ |
| 2 | 0 | -1 | $\frac{1}{2}$ | 1 | $\frac{5}{4}$ |
| 3 | 0 | 0 | $\frac{5}{4}$ | $\frac{1}{2}$ | $\frac{21}{8}$ |
| 4 | 0 | 0 | $\frac{21}{8}$ | $\frac{5}{4}$ | $\frac{85}{16}$ |
| 5 | 0 | 0 | $\frac{85}{16}$ | $\frac{21}{8}$ | $\frac{341}{32}$ |

b) $Y(z) = X(z) - X(z)z^{-1} + \frac{5}{2}Y(z)z^{-1} - Y(z)z^{-2}$

$$H(z) = \frac{1 - z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1 - z^{-1}}{(1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})}$$




c) No

2. (continued)

d) $$H(z) = \frac{\cancel{\phantom{x}}\ 2/3}{1 - 2z^{-1}} + \frac{1/3}{1 - \frac{1}{2}z^{-1}}$$

For causal system, ROC ~~[illegible]~~ is $|z| > 2$

$\Rightarrow$ both terms are right-sided

From tables

$$h[n] = \frac{2}{3}\, 2^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^n u[n]$$

3. (25 pts.) A 30 sec. segment of speech is low pass filtered to bandlimit it to 4 kHz, then sampled at 8 kHz. This speech signal is corrupted by noise in the band of frequencies from 50 Hz to 100 Hz.

You are going to compute the DFT $X[k]$ of the signal $x[n]$, multiply by an appropriate filter frequency response $H[k]$ to cut out the corrupted portion of the signal spectrum, and then compute the inverse DFT to obtain the cleaned-up signal $y[n]$.

a. (5) Choose the minimum length $N$ of the DFT that will allow you to use a radix 2 FFT algorithm.

b. (15) Specify exactly what the frequency response $H[k]$ should be.

c. (5) Assume that your processor can execute 1 million complex floating point operations each second and that this includes overhead for indexing, memory fetches. Compute the amount of time required to process the entire signal.

a) No. samples $N_s = 30 \times 8000 = 24{,}000$ 2

$N = 2^{15} \cong 32{,}000 > 24{,}000$ is min. length DFT 3

exact $N = 32768$

b) cutoff frequencies are

$2\pi\left(\frac{50}{8000}\right) = \frac{\pi}{80}$ rad/sample & $2\pi\left(\frac{100}{8000}\right) = \frac{\pi}{40}$ rad/sample

solve for DFT index values $k_1$ & $k_2$ ($k_2 \cong 2k_1$)

$2\pi \frac{k_1}{N} = \frac{\pi}{80} \Rightarrow k_1 = \frac{N}{160} = \frac{2^{15}}{2^5 \cdot 5} = \frac{2^{10}}{5} = \frac{1024}{5} = \cancel{200.8}$ 204.8 Round down.

Rounding to nearest integer, get $k_1 = \cancel{201}$ 205 round up.

Similarly, $k_2 = \cancel{402}$ 409.6 ≈ 410

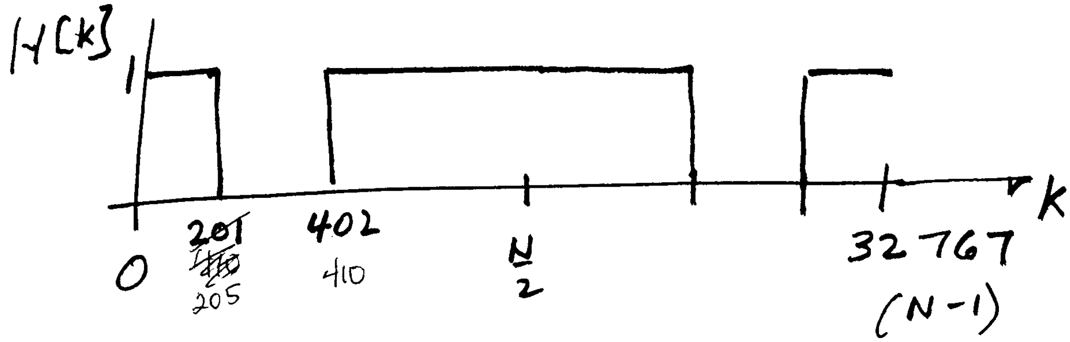
A more conservative strategy would be to choose $k_1 = \cancel{200}$ 204 & $k_2 = \cancel{402}$ 410

Also need stopband from $k_3$ to $k_4$ where

$k_4 = N - k_1 = 32768 - \cancel{201}$ 205 $= 325\cancel{67}$ 63 (64)

$k_3 = N - k_2 = 32768 - \cancel{402}$ 410 $= 323\cancel{66}$ 58 (32358)

3. (continued)




$$|H[k]| = \begin{cases} 0, & 201 \le k \le 402, \\ & \cancel{32366} < k \le \cancel{32567} \\ & \;\; 32358 \qquad\quad 32563 \\ 1, & \text{else} \end{cases}$$

c) Assume ~~appx~~ "complex operation" = 1 complex multiply + 1 complex add

Need to compute forward DFT ($N \log_2 N$ COs), multiply pt. by pt. with $H[k]$ ($N/2$ COs) (since no adds), then take inverse DFT ($N \log_2 N$)

Total is $2N \log_2 N + \frac{N}{2} \cong 15 \cdot 64 \times 10^3 + 16 \times 10^3$
$= 976 \times 10^3 \approx 10^6$ CO's

$\Rightarrow T = 1$ sec.

Summary of corrected sol.

a) $N_s = 30 \times 8000 = 240,000$ (2)

$2^{17} = 131072$ won't work.

$2^{18} = 262144$ $\qquad$ $2^{18}$ is min. (3)

done

b)

$\omega_1 = 2\pi \frac{50}{8000} = \frac{\pi}{80}$ rad/sample (3)

$\omega_2 = 2\pi \frac{100}{8000} = \frac{\pi}{40}$ rad/sample (3)

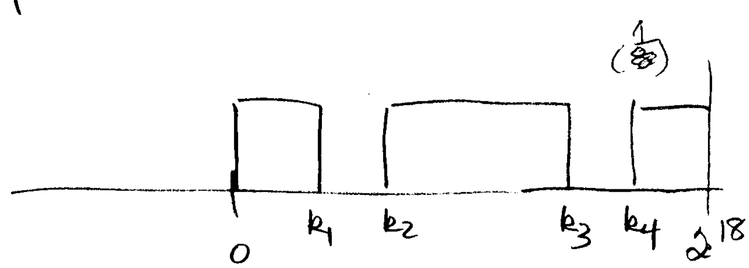
$k_1 = \frac{f}{f_s} N = 1638.4$ $\quad$ (round down) (2)
(1638)

$k_2 = \frac{f}{f_s} N = 3276.8$ $\quad$ round up. (2)
(3277)

$k_3 = 2^{18} - k_2 = 258,867$ (2)

$k_4 = 2^{18} - k_1 = 260506$ (2)

/5




c) $2(N \log_2 N) + \frac{N}{2} = \overset{(1)}{2}\overset{(2)}{\left(2^{18} \cdot 18\right)} + \overset{(\cancel{2})(1)}{\frac{2^{18}}{2}} =$

$= 2(4718592) + 131072$

$= 9,568,256$ complex Ops.

$\Rightarrow$ 9.6 seconds $(\cancel{1})$

4. (25 pts.) Consider the DT signal

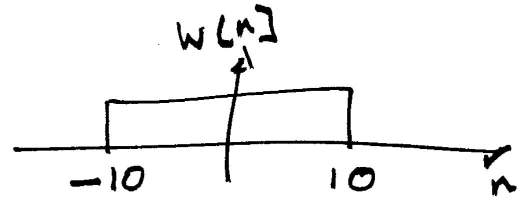
$$x[n] = \begin{cases} \cos(\pi n/2), & n<0 \\ \cos(\pi n/6), & n \geq 0 \end{cases}$$

Suppose we define the STDTFT as

$$X(e^{j\omega}, n) = \sum_k x[k] w[n-k] e^{-j\omega k},$$

where the window function is given by

$$w[n] = \begin{cases} 1, & -10 \leq n \leq 10 \\ 0, & \text{else} \end{cases}$$

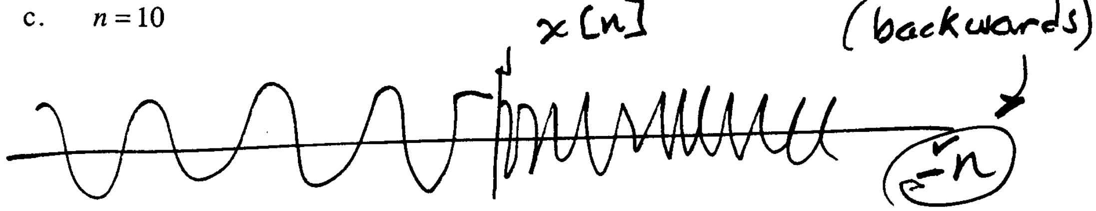



Calculate the STDTFT and sketch its magnitude for the following three cases:

a. $n < -10$

b. $n > 10$

c. $n = 10$




a) window covers only $\cos(\pi n/2)$

b) window covers only $\cos(\pi n/6)$

let $w[n-k] = w[k-n] \xleftrightarrow{DTFT} W_n(e^{j\omega})$

also $\cos(\omega_0 n) \xleftrightarrow{DTFT} \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$, $|\omega| < \pi$

circular conv. w/ $\frac{1}{2\pi}$

then

$$X(e^{j\omega}, n) \triangleq \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] * W_n(e^{j\omega}), \quad |\omega|<\pi$$

$$= \frac{1}{2}\left[W_n(e^{j(\omega-\omega_0)}) + W_n(e^{j(\omega+\omega_0)})\right], \quad |\omega|<\pi$$

$$W_n(e^{j\omega}) = \sum_{k=n-10}^{n+10} e^{-j\omega k} = \sum_{k=0}^{20} e^{-j\omega[l+(n-10)]}$$

$l = k-(n-10)$

$$= e^{-j\omega(n-10)} \sum_{k=0}^{20} e^{-j\omega l}$$

4. (continued)

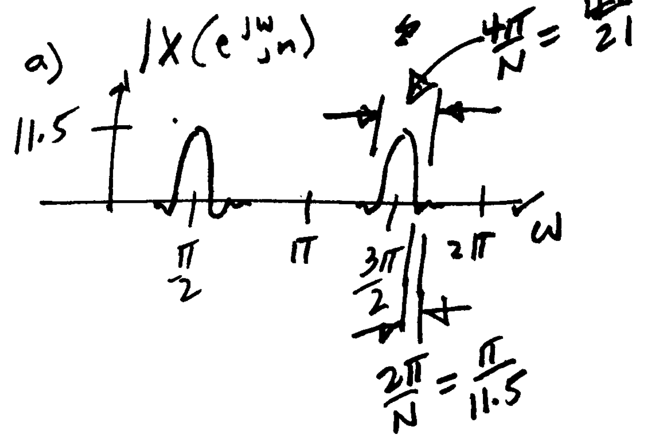
$$W_n(e^{j\omega}) = e^{-j\omega(n-10)} \frac{1-e^{-j\omega 21}}{1-e^{-j\omega}}$$

$$= e^{-j\omega(n-10)} \, e^{-j\omega(\underbrace{\frac{21}{2}-\frac{1}{2}}_{10})} \frac{\sin(\omega 10.5)}{\sin(\omega 0.5)}$$

$$= e^{-j\omega n} \frac{\sin(21\omega/2)}{\sin(\omega/2)}$$

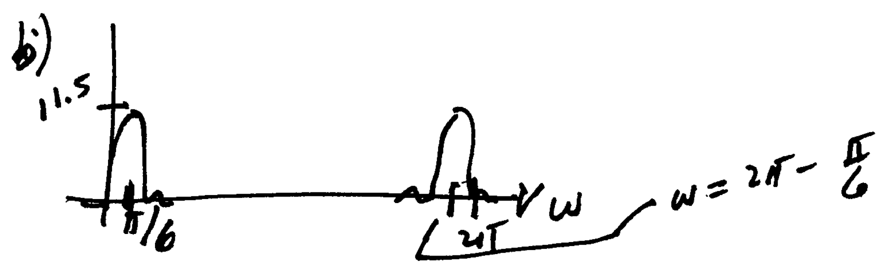
$$X(e^{j\omega}, n) \approx \frac{1}{2}\left\{ e^{-j(\omega-\omega_0)n} \frac{\sin[21(\omega-\omega_0)/2]}{\sin[(\omega-\omega_0)/2]} + e^{-j(\omega+\omega_0)n} \frac{\sin[2\pi(\omega+\omega_0)/2]}{\sin[(\omega+\omega_0)/2]} \right\}$$

a) $|X(e^{j\omega}, n)|$

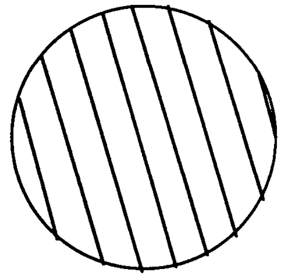



c) same as a)
since $\cos(\frac{\pi}{2}n) = \cos(\frac{\pi}{6}n)$
at $n=0$

b)



5. (25 pts.) Consider the 2-D signal $f(x,y)$ shown below



which consists of a circle of radius 5 centered at the origin. Within the circle, $f(x,y)$ is given by

$$\tfrac{1}{2}+\tfrac{1}{2}\cos[2\pi(x+y/5)].$$

Outside the circle, $f(x,y)=0$. The lines inside the circle in the figure above approximately represent lines of constant amplitude in the sinewave.

a. (17) Using standard transform relations and standard transform pairs, find a simple expression for the continuous-space Fourier transform (CSFT) $F(u,v)$ of this signal.

b. (8) Sketch what $F(u,v)$ looks like.

a) $$f(x,y) = \left\{\tfrac{1}{2} + \tfrac{1}{2}\cos\left[2\pi(x+y/5)\right]\right\} \operatorname{circ}\left(\frac{x}{10}, \frac{y}{10}\right)$$

$$F(u,v) = \left\{\tfrac{1}{2}\,\delta(u,v) + \tfrac{1}{2}\,\delta\left(u-1, v-\tfrac{1}{5}\right) + \tfrac{1}{2}\,\delta\left(u+1, v+\tfrac{1}{5}\right)\right\}$$

$$** \; 100 \operatorname{jinc}(10u, 10v)$$

$$= 50\left\{\operatorname{jinc}(10u, 10v) + \operatorname{jinc}\left(10(u-1), 10\left(v-\tfrac{1}{5}\right)\right) + \operatorname{jinc}\left(10(u+1), 10\left(v+\tfrac{1}{5}\right)\right)\right\}$$

b)