October 20, 1999 **Name:** Solution

EE 438 **Exam No. 2** **Fall 1999**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the following Z transform

$$X(z) = \frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}$$

a. (5 pts.) Find all possible regions of convergence for $X(z)$.

b. (20 pts.)For each region of convergence that you identified in part a), find the inverse Z transform $x[n]$.

Factor polynomial

$$1+\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2} = (1+z^{-1})(1-\frac{1}{2}z^{-1})$$

PFE

$$\frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}} = \frac{A}{(1+z^{-1})} + \frac{B}{(1-\frac{1}{2}z^{-1})}$$

poles at $z=-1$, $z=\frac{1}{2}$

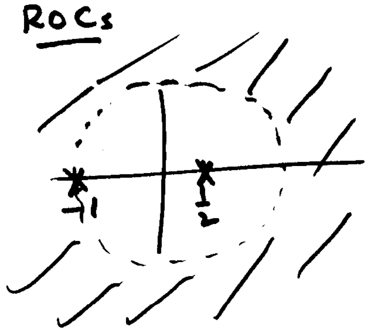
$$\Rightarrow \quad 1+\frac{1}{2}z^{-1} = A - \frac{A}{2}z^{-1} + B + Bz^{-1}$$

$$\Rightarrow \quad A+B=1 \qquad B-A/2 = 1/2 \quad \Rightarrow A = 2B-1$$

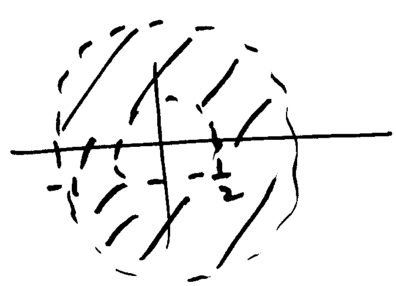
$$\Rightarrow \quad 3B = 2 \text{ or } B = 2/3 \qquad A = 1/3$$

check $\quad 1/3 - 1/6\, z^{-1} + 2/3 + 2/3\, z^{-1} = 1 + \frac{1}{2}z^{-1}$ ✓

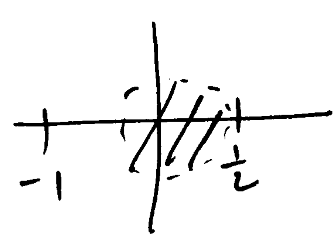
a) ROCs



I. $|z|>1$



II. $\frac{1}{2}<|z|<1$



III. $|z|<\frac{1}{2}$

1. (continued)

b) Using formulas

I. Both terms are r.s.

$$x[n] = \frac{1}{3}(-1)^n u[n] + \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

II. First term l.s., second term r.s.

$$x[n] = -\frac{1}{3}(-1)^n u[-n-1] + \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

III. Both terms are l.s.

$$x[n] = -\frac{1}{3}(-1)^n u[-n-1] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

2. (25 pts.) Consider the signal

$$x(t) = \cos(2\pi(50)t) + 2\cos(2\pi(7000)t)$$

which is sampled starting at time $t = 0$ with an ideal sampler (no prefilter) operating at a 10 kHz rate. We then compute the DFT X[k], k = 0, …, 99, of the first 100 samples of this signal.

a. (15 pts.) Determine the approximate values for k (in the range 0 to 99) at which you will observe spectral peaks, and give the approximate amplitude of these peaks.

b. (5 pts.) For each spectral peak that you found in part a), state whether or not that peak will exhibit leakage and picket fence effect.

c. (5 pts.) Sketch the magnitude of the DFT spectrum, clearly showing the effect of leakage and picket fence, wherever it is present.

Note: You do not need to calculate the exact DFT to answer this part (or any part of this problem, for that matter). You only need to show clearly that you know what leakage and picket fence effect look like.

a) $T = 10^{-4}$

$$x[n] = \cos(2\pi(50)n10^{-4}) + 2\cos(2\pi(7000)n10^{-4})$$

$$= \cos\left(\frac{\pi}{100}n\right) + 2\cos(1.4\pi n)$$

$\omega_{d_1} = 0.01\pi$ note that this exceeds π rad/sample

$\therefore \omega_{d_2} = 0.6\pi$

DFT generates samples of spectra at $\frac{2\pi k}{100}$

To find k values, solve $\frac{2\pi k}{100} = \omega_d \Rightarrow k = \frac{\omega_d 100}{2\pi}$

$k_1 = \frac{0.01\pi \, 100}{2\pi} = \frac{1}{2}$ so peak is midway between $k = 0$ & $k = 1$; also get peak at $k = 99$ & $k = 100$ (same as $k=0$)

$(100-1)$

$k_2 = \frac{0.6\pi \, 100}{2\pi} = 30$ peak falls right on sample

also get peak at $k_2 = 100 - 30 = 70$

2. (continued)

From formulas, $\cos(2\pi k_0 n/N) \leftrightarrow \frac{N}{2}\{\delta(k-k_0) + \delta(k-(N-k_0))\}$

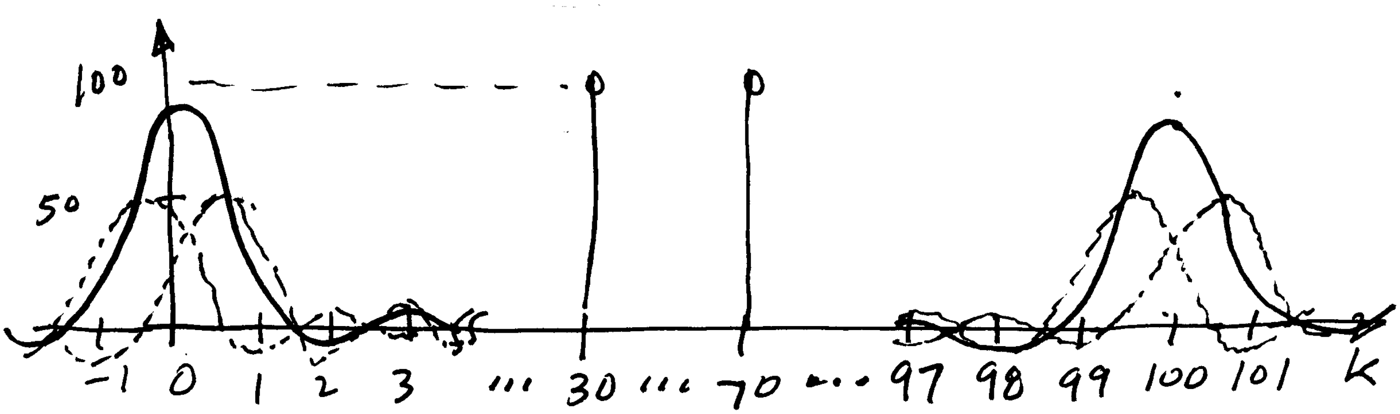
$\Rightarrow$ amplitude for peak at $k = 0, 1, 99, 100 \approx 50$

amplitude for peak at $k = 30, 70 \approx 100$

b) $k = 0, 1, 99, 100$ shows leakage & picket fence

$k = 30, 70$ shows no leakage or picket fence

c) $X[k]$



take absolute value to get magnitude

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% Program to compute exact answer for Problem 2.
%
n = 0:1:99;
T = 10^(-4);
x = cos(2.*pi.*50.*n.*T) + 2.*cos(2.*pi.*7000.*n.*T);
subplot(2,1,1)
stem(n,x)
xlabel('n')
ylabel('x[n]')
X = abs(fft(x));
subplot(2,1,2)
stem(n,X)
xlabel('k')
ylabel('X[k]')
gtext('Note that X[0] = 0, rather than the value of 70 or so predicted by approximate closed form soluti
gtext('This is because of phase factors ignored in sketching the magnitude of the DFT.')
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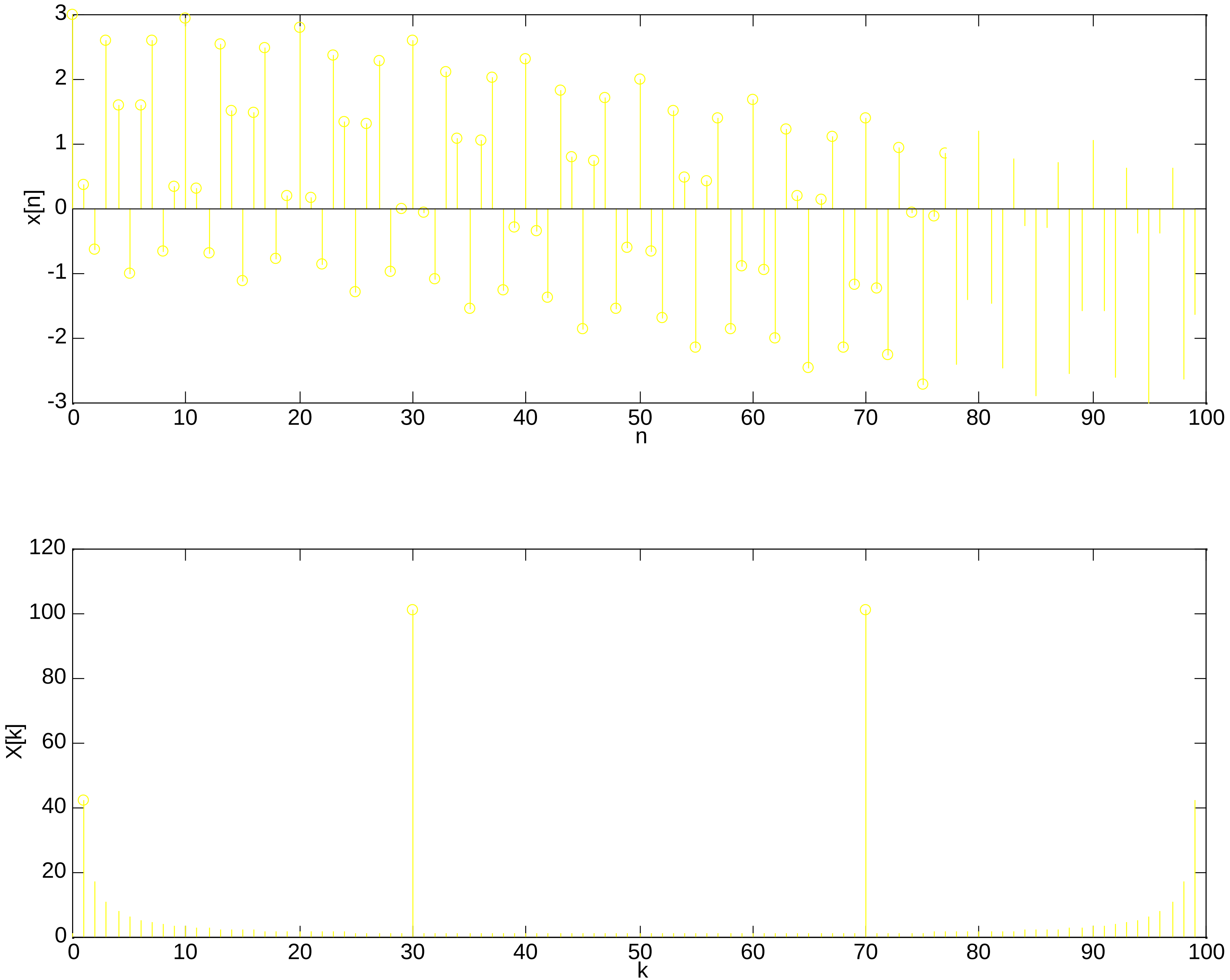
3
2
1
0
-1
-2
-3
x[n]
0
10
20
30
40
50
60
70
80
90
100
n
120
100
80
60
40
20
0
X[k]
0
10
20
30
40
50
60
70
80
90
100
k

3. (25 pts.) Let x[n], n = 0,…,N-1 be an N point signal with N pt. DFT $X^{(N)}[k]$, k = 0,…, N-1. Define a new 10N point signal y[n] by repeating x[n] 10 times, i.e. y[n] = x[n mod N], n = 0, …, 10N-1.

a. (23 pts.) Find a simple expression for the 10N point DFT $Y^{(10N)}[k]$, k = 0, …, 10N-1 in terms of the N point DFT $X^{(N)}[l]$, l = 0,…, N-1.

b. (2 pts.) Compare your results to the CTFT of a periodic signal.

a) $$Y^{(10N)}[k] = \sum_{n=0}^{10N-1} y[n] e^{-j 2\pi nk/10N}$$

let $$y[n] = \frac{1}{N} \sum_{l=0}^{N-1} X[l] e^{j 2\pi nl/N}$$

note that this is periodic in n with period N

$$Y^{(10N)}[k] = \frac{1}{N} \sum_{l=0}^{N-1} X[l] \sum_{n=0}^{10N-1} e^{j 2\pi(10l - k)n/10N}$$

$$\sum_{n=0}^{10N-1} e^{j 2\pi(10l - k)n/10N} = \begin{cases} 10N, & k = 10l \\ \dfrac{1 - e^{j 2\pi(10l-k)}}{1 - e^{j 2\pi(10l-k)}} = 0, & \text{else} \end{cases}$$

$$\therefore Y^{(10N)}[k] = 10 \sum_{l=0}^{N-1} X[l] \delta[k - 10l]$$

$$= \begin{cases} 10 X[k/10], & k/10 = \text{integer} \\ 0, & \text{else} \end{cases}$$

b) CTFT of periodic signal has spectral energy only at multiples of fundamental frequency. This DFT has spectral energy only at k is a multiple of 10.

4. (25 pts.) You have an ASIC (Application-Specific Integrated Circuit) module for computing a 16 point FFT.
   a. (12 pts.) Derive the equations that show how an 80 point DFT may be expressed in terms of 16 point DFTs
   b. (8 pts.) Draw a complete flow diagram for your 80 point DFT, using the 16 point FFT as a black box simply labeled as "16 point FFT" without showing any details of how it is computed. Be sure to label all twiddle factors, and show clearly how the data is ordered at the input and output of the flow diagram.
   c. (5 pts.) Determine how many complex operations are required to compute your 80 point DFT,

a) $80 = 5 \cdot 16 \Rightarrow$ break signal into 5 length 16 signals. Let $n = 5m + l$

$$X^{(80)}[k] = \sum_{l=0}^{4} \sum_{m=0}^{15} x[5m+l] e^{-j2\pi(5m+l)k/80}$$

$$= \sum_{l=0}^{4} e^{-j2\pi lk/80} \underbrace{\sum_{m=0}^{15} \underbrace{x[5m+l]}_{x_l[m]} e^{-j2\pi mk/16}}_{\text{16 pt. DFT of } x_l[m],\ m=0,\ldots,15}$$

So $$X^{(80)}[k] = \sum_{l=0}^{4} e^{-j2\pi lk/80} X_l^{(16)}[k] \quad (*)$$

c) i) Have to compute 5 16 pt. ~~DFTs~~ $\Rightarrow 5 \cdot \underbrace{16 \cdot 4}_{N\log_2 N} = 320$ c.o.

c.o. = complex operation

ii) To do $(*)$ requires another $5 \cdot 80 = 400$ c.o.

So total is 720 c.o.

4. (continued)

Implementation of (*)

(b)