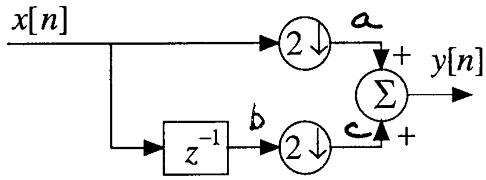
December 16, 1998 **Name:** Solution

EE 438 **Final Exam** **Fall 1998**

- You have 120 minutes to work the following six problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Multirate systems. Consider the system shown below.



a. (2) Circle the correct choice for the system properties (no proofs needed, no partial credit will be given for this part only):
   i. The system is linear/nonlinear.
   ii. The system is time-invariant/time-varying.

b. (14) Find a simple expression for the DTFT $Y(\omega)$ of the output in terms of the DTFT $X(\omega)$ of the input.

c. (9) Draw the block diagram for an equivalent system that consists of a digital filter and just one down-sampler. Specify the difference equation for the digital filter.

b) $Y(\omega) = X_a(\omega) + X_c(\omega)$

$$X_a(\omega) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\omega - 2\pi k}{2}\right) = \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega-2\pi}{2}\right)\right]$$

$$X_c(\omega) = \frac{1}{2}\left[X_b\left(\frac{\omega}{2}\right) + X_b\left(\frac{\omega-2\pi}{2}\right)\right]$$

$$X_b(\omega) = e^{-j\omega}X(\omega)$$

Combining

$$Y(\omega) = \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega-2\pi}{2}\right)\right] + \frac{1}{2}\left[e^{-j\frac{\omega}{2}}X\left(\frac{\omega}{2}\right) + e^{-j\left(\frac{\omega-2\pi}{2}\right)}X\left(\frac{\omega-2\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\left(1+e^{-j\frac{\omega}{2}}\right)X\left(\frac{\omega}{2}\right) + \left(1+e^{-j\left(\frac{\omega-2\pi}{2}\right)}\right)X\left(\frac{\omega-2\pi}{2}\right)\right]$$

1. (continued)

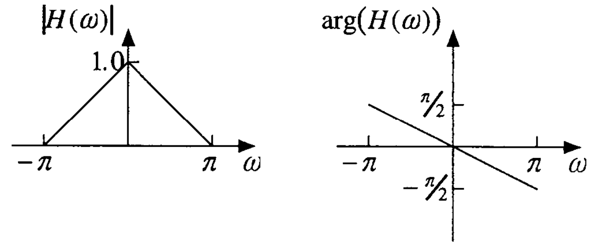
c)

$$x[n] \longrightarrow \boxed{\text{Filter}} \xrightarrow{v[n]} (2\downarrow) \longrightarrow y[n]$$

$$v[n] = x[n] + x[n-1]$$

2. (25 pts.) Frequency response and sampling. Consider the system shown below where the A/D and D/A convertors operate at a sampling rate of 1 kHz and are both ideal.

a. (15) Find the output $y(t)$ when $x(t) = \cos(2\pi(200)t)$ and the digital filter has the following magnitude and phase frequency response.



b. (10) Find the output $y(t)$ when $x(t) = \sin(2\pi(900)t)$ and the digital filter is allpass with no phase delay.

a) Nyquist frequency is 500 Hz ⇒ no aliasing for 200 Hz

$$x[n] = x(nT) = x(10^{-3} n) = \cos\left(\frac{2\pi}{5} n\right)$$

$$X(\omega) = \frac{1}{2}\left[\delta\left(\omega - \frac{2\pi}{5}\right) + \delta\left(\omega + \frac{2\pi}{5}\right)\right] \quad |\omega| < \pi$$

$$Y(\omega) = H(\omega) X(\omega)$$

$$= \frac{1}{2}\left[H\left(\frac{2\pi}{5}\right)\delta\left(\omega - \frac{2\pi}{5}\right) + H\left(-\frac{2\pi}{5}\right)\delta\left(\omega + \frac{2\pi}{5}\right)\right]$$

$$= \frac{1}{2}\left[\frac{3}{5} e^{-j\frac{\pi}{5}} \delta\left(\omega - \frac{2\pi}{5}\right) + \frac{3}{5} e^{j\frac{\pi}{5}} \delta\left(\omega + \frac{2\pi}{5}\right)\right]$$

$$y[n] = \frac{1}{2}\left[\frac{3}{5} e^{-j\frac{\pi}{5}} e^{+j\frac{2\pi}{5} n} + \frac{3}{5} e^{j\frac{\pi}{5}} e^{-j\frac{2\pi}{5} n}\right]$$

$$= \frac{3}{5} \cos\left(\frac{2\pi}{5} n - \frac{\pi}{5}\right)$$

$$= \frac{3}{5} \cos\left(\frac{2\pi}{5}\left(n - \frac{1}{2}\right)\right)$$

2. (continued)

$$y[n] = \frac{3}{5}\cos\left(\frac{2\pi(200)n}{1000} - \frac{\pi}{5}\right)$$

$$\therefore\ y(t) = \frac{3}{5}\cos\left(2\pi(200)t - \frac{\pi}{5}\right)$$

$$= \frac{3}{5}\cos\left(2\pi(200)\left(t - \frac{1}{2000}\right)\right)$$

b) Two answers are possible:

i) Ideal A/D includes LPF with cutoff at 500 Hz $\Rightarrow y(t) = 0$

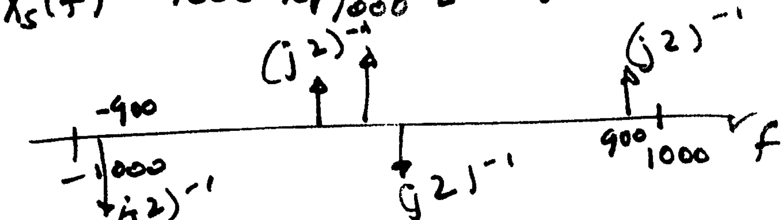
ii) Ideal A/D does not include any LPF (It really should).

$$x(t) = \sin(2\pi(900)t)$$

$$= \frac{1}{j2}\left[e^{+j2\pi(900)t} - e^{-j2\pi(900)t}\right]$$

$$X(f) = \frac{1}{j2}\left[\delta(f-900) - \delta(f+900)\right]$$

$$X_s(f) = 1000\ \mathrm{rep}_{1000}[X(f)]$$



Ideal D/A with LPF passes only

$$Y(f) = \frac{1}{j2}\left[-\delta(f-100) + \delta(f+100)\right]$$

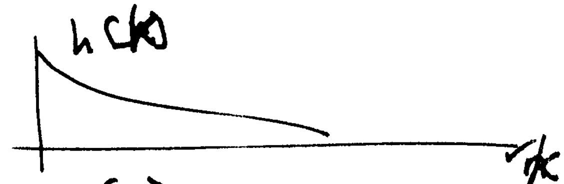
$$\Rightarrow y(t) = -\sin[2\pi(100)t]$$

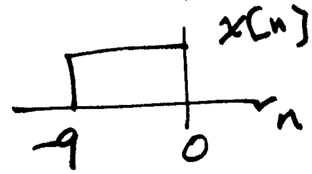
3. (25 pts.) Filtering and convolution. A digital filter that is linear, time-invariant, and causal has impulse response $h[n] = 2^{-n}u[n]$.

   a. (15) Use *convolution* to find the response $y[n]$ to the input signal

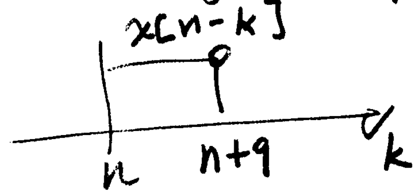
$$x[n] = \begin{cases} 1, & -9 \le n \le 0, \\ 0, & \text{else.} \end{cases}$$

   b. (10) Find a difference equation that can be used to implement this filter.





$$y[n] = \sum_k h[k]\, x[n-k]$$



a)

I. $n \le -10$ $\quad y[n] = 0$

II. $-9 \le n \le 0$ $\quad y[n] = \sum_{k=0}^{n+9} 2^{-k}$

$$= \frac{1 - 2^{-(n+10)}}{1 - 2^{-1}}$$

$$= 2\left[1 - 2^{-(n+10)}\right]$$

III. $n \ge 0$

$$y[n] = \sum_{k=n}^{n+9} 2^{-k} = \sum_{\ell=0}^{9} 2^{-(\ell+n)} = 2^{-n} \sum_{\ell=0}^{9} 2^{-\ell}$$

$\ell = k - n$

$k = \ell + n$

$$= 2^{-n}\, \frac{1 - 2^{-10}}{1 - 2^{-1}}$$

$$\approx 2^{-(n-1)}$$

3. (continued)

b) $h[n] = 2^{-n} u[n]$

$$H(z) = \frac{1}{1 - 2^{-1} z^{-1}} \qquad |z| > 2^{-1}$$

$$Y(z) = H(z) X(z)$$

$$\Rightarrow Y(z) = X(z) + 2^{-1} Y(z) z^{-1}$$

$$\Rightarrow y[n] = x[n] + \tfrac{1}{2} x[n-1]$$

4. (25 pts.) DFT. You have a subroutine that will compute the radix 2 FFT for length $N = 2^M$, for any integer M. You need to write a subroutine that will compute an exact length 96 DFT (no zero-padding allowed).
   a. (13) Derive an efficient algorithm for computing the 96 point DFT that uses the radix 2 FFT subroutine as a component.
   b. (8) Draw a complete block diagram of your algorithm, showing all twiddle factors, and showing the radix 2 FFT simply as a component *without* any details of what is inside it.
   c. (4) Find the number of complex operations required *per output data value* for your algorithm.

$96 = 3 \cdot 32$

$$X^{96}[k] = \sum_{n=0}^{95} x(n)\, e^{-j 2\pi nk/96}$$

$$= \sum_{m=0}^{31} x[3m]\, e^{-j 2\pi (3m)k/96} + \sum_{m=0}^{31} x[3m+1]\, e^{-j 2\pi (3m+1)k/96} + \sum_{m=0}^{31} x[3m+2]\, e^{-j 2\pi (3m+2)k/96}$$

$$= \sum_{m=0}^{31} x[3m]\, e^{-j 2\pi mk/32} + e^{-j 2\pi k/96} \sum_{m=0}^{31} x[3m+1]\, e^{-j 2\pi mk/32} + e^{-j 2\pi k(2)/96} \sum_{m=0}^{31} x[3m+2]\, e^{-j 2\pi mk/32}$$

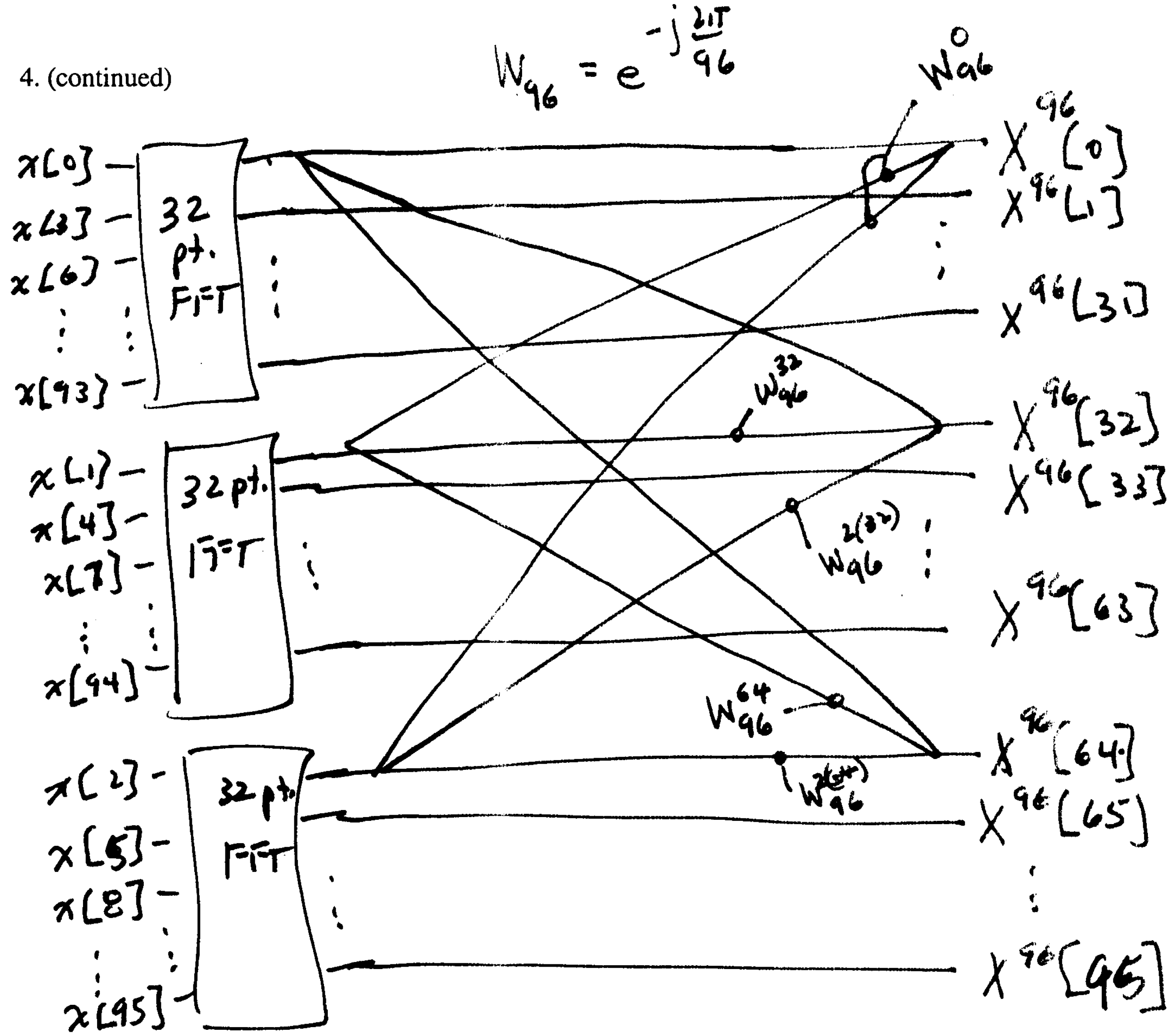
$$= X_0^{32}[k] + e^{-j 2\pi k/96} X_1^{32}[k] + e^{-j 2\pi k(2)/96} X_2^{32}[k]$$

$$x_i[m] = x[3m+i], \quad i = 0, 1, 2$$
$$m = 0, 1, \ldots, 31$$

4. (continued)

$W_{96} = e^{-j\frac{2\pi}{96}}$

b)



c) Total No. C.O.

3 x 32 pt FFT $\quad 3 \cdot 32 \log_2(32)$

Combine results $\quad \sim 3 \cdot 96$

No. C.O./output pt. $\cong$ 8

5. (25 pts.) Modeling of speech signals. In class, we discussed linear predictive coding as a means of efficiently representing the speech waveform using the parameters of a model rather than samples from the speech waveform itself. In this problem, we will consider a different type of model for a voiced epoch of speech. We will work in discrete-time. Recall that a voiced phoneme $s[n]$ can be represented by the vocal tract response $v[n]$ repeated at an interval corresponding to the pitch period N; so our model $\hat{s}[n]$ is given by

$$\hat{s}[n] = \sum_k v[n-kN]$$

We will model the vocal tract response as a summation of exponentials (for simplicity we'll just use two terms here); so we have

$$v[n] = a p^n u[n] + b q^n u[n],$$

where a, b, p, q are constants which may be complex-valued, in which case $a^* = b$ and $p^* = q$.

Assume that $a = e^{j\pi/3}$ and $p = \frac{1}{2}e^{j\pi/4}$.

a. (10) Find a simple expression for $v[n]$, and sketch the resulting waveform.

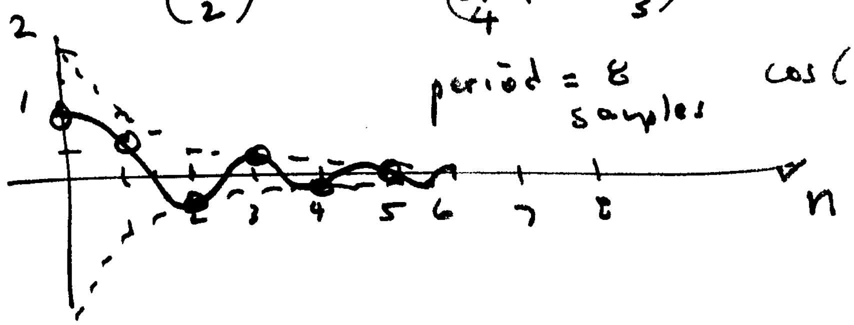
b. (10) Find the Z transform $V(z)$ of the vocal tract response, and express as a ratio of polynomials in z. Be sure to state the region of convergence.

c. (4) Plot the poles and zeros of $V(z)$.

d. (1) What is the formant frequency for this phoneme (in radians/sample)?

a) $v[n] = \left[e^{j\pi/3}\left(\frac{1}{2}e^{j\pi/4}\right)^n + e^{-j\pi/3}\left(\frac{1}{2}e^{-j\pi/4}\right)^n\right]u[n]$

$= \left(\frac{1}{2}\right)^{n-1}\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)u[n]$



period = 8 samples $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

b) $V(z) = \dfrac{a}{1-pz^{-1}} + \dfrac{b}{1-qz^{-1}}$ $|z| > \max[p, q]$

$= \dfrac{a(1-qz^{-1}) + b(1-pz^{-1})}{(1-pz^{-1})(1-qz^{-1})}$

$= \dfrac{a+b-(aq+bp)z^{-1}}{(1-pz^{-1})(1-qz^{-1})}$

5. (continued)

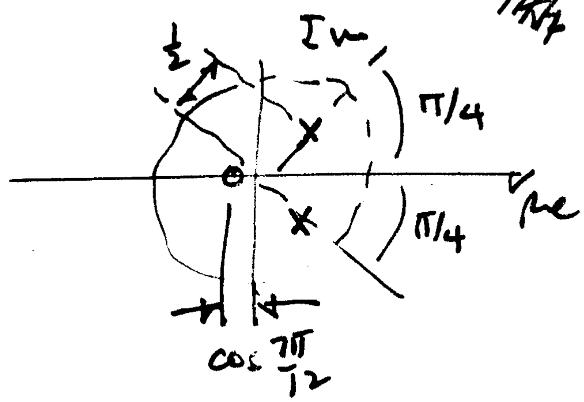
c) poles p, q $\left(\frac{1}{2}e^{\pm j\pi/4}\right)$

zero $z = \dfrac{aq + bp}{a + b}$

$$= \frac{1}{2}\,\frac{e^{j(\pi/3 + \pi/4)} + e^{-j(\pi/3 + \pi/4)}}{e^{j\pi/3} + e^{-j\pi/3}}$$

$$= \frac{1}{2}\,\frac{\cos(7\pi/12)}{\cos(\pi/3)} = \cos\left(\frac{7\pi}{12}\right)$$

$\approx$ [illegible]



d) $\pi/4$ rad./sample

6. (25 pts.) Spatial filtering. Consider the image $x[m,n]$ shown below:

$$\begin{array}{c} \xrightarrow{n} \\ m \downarrow \end{array} \quad \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{matrix}$$

Suppose we filter this image with a filter that has impulse response:

$h[m,n]$

m \ n	-1	0	1
-1	$-1/9$	$-1/9$	$-1/9$
0	$5/9$	$5/9$	$5/9$
1	$-1/9$	$-1/9$	$-1/9$

a. (9) Compute the output image $y[m,n]$.

b. (1) Is the filter DC preserving?

c. (3) Find the separable components $h_1[m]$ and $h_2[n]$, such that $h[m,n] = h_1[m]h_2[n]$.

d. (5) Find a simple expression for the magnitude of the frequency response $|H(\mu,\nu)|$ for this filter.

e. (2) Sketch $|H(\mu,\nu)|$ for the two cases:
  i. $\mu = 0$,
  ii. $\nu = 0$,

f. (1) Discuss the relation between your answers to parts a., b., and e.

g. (4) Since the filter is separable, $y[m,n]$ may be computed in either one of two ways:
  i. Direct 2-D convolution using $h[m,n]$ given above.
  ii. 1-D convolution along columns with $h_1[m]$ followed by 1-D convolution along rows with $h_2[n]$.

  How many operations *per output pixel* are required for each approach?

a) $y[m,n] = \sum_k \sum_l h[m-k, n-l]\, x[m,n]$

Due to symmetry, flipping h will have no effect.
Handle boundaries by extending pixels from edges

6. (continued)

$$
y[m,n] \quad
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/9 & -2/9 & -3/9 & -3/9 & -3/9 \\
0 & 4/9 & 8/9 & 12/9 & 12/9 & 12/9 \\
0 & +3/9 & 6/9 & 1 & 1 & 1 \\
0 & +3/9 & 6/9 & 1 & 1 & 1 \\
0 & +3/9 & 6/9 & 1 & 1 & 1
\end{array}
$$

b) Yes

$$\sum_m \sum_n h[m,n] = 1$$

c) $h_1[m] = [-1/3 \quad 5/3 \quad -1/3]$

$h_2[n] = [1/3 \quad 1/3 \quad 1/3]$

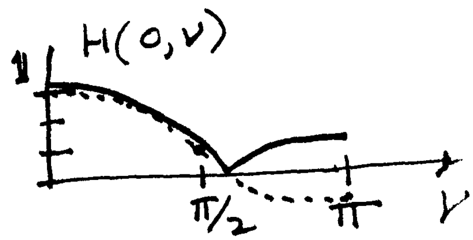
d) $H(\mu,\nu) = H_1(\mu) H_2(\nu)$, $\quad H_1(\mu) = \sum_m h_1[m] e$

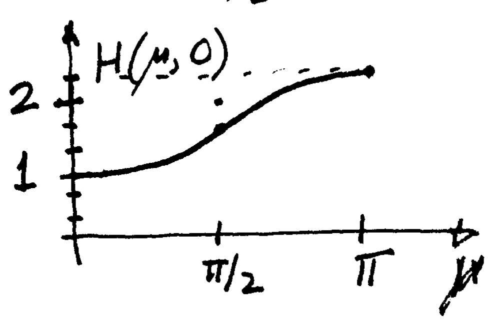
$H_1(\mu) = 5/3 - 2/3 \cos\mu$

$H_2(\nu) = 1/3 + 2/3 \cos\nu$

$\therefore \quad |H(\mu,\nu)| = |5/3 - 2/3 \cos\mu| \, |1/3 + 2/3 \cos\nu|$

e)





f) DC preserving $\Leftrightarrow$ $H(0,0) = 1$

Spatial domain:
- sharpen ~~vertical~~ horizontal edge
- blur vertical edge

frequency domain:
- low pass along ν
- high emphasis along ~~ν~~ μ

g)
1. i) 9 multiplies/output pt. 8 adds/output pt.
2. (for general 3×3
3. filter kernel)
4.
5. ii) 2 × 3 multiplies/output pt.
6. 2 × 2 adds/output pt.