November 23, 1998 **Name:** SOLUTIONS

EE 438 **Exam No. 3** **Fall 1998**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the random process defined by

$$X[n] = 2\cos\left(\tfrac{\pi}{2}n + \theta\right)$$

where θ is a random variable uniformly distributed on the interval $[0,2\pi)$

a. (8) Find the expected value of X[n].

b. (8) Find the variance of X[n].

c. (8) Find the autocorrelation $r_{XX}[m,n] = E\{X[m]X[n]\}$ of this process.

d. (1) Is X[n] wide-sense stationary?

Note: the following formula may be useful.

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

a) $E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

$E\{X[n]\} = \int_0^{2\pi} 2\cos\left(\frac{\pi}{2}n + \theta\right)\frac{1}{2\pi}\,d\theta$

$= \frac{1}{\pi}\sin\left(\frac{\pi}{2}n+\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = \underline{\underline{0}}$

b) $E\{(X[n]-\overline{X[n]})^2\} = E\{(X[n])^2\}$

$= \int_0^{2\pi} 4\cos^2\left(\frac{\pi}{2}n+\theta\right)\frac{1}{2\pi}\,d\theta$ Using the formula.

$= \frac{2}{\pi}\int_0^{2\pi}\frac{1}{2}\left(\cos(0)^{\nearrow 1} + \underbrace{\cos\left(2\left(\frac{\pi}{2}n+\theta\right)\right)}_{\text{Integrates to zero}}\right)d\theta$

$= \frac{1}{\pi}\cdot 2\pi = \underline{\underline{2}}$

1. (continued)

c) $r_{xx}[m,n] = E\{2\cos(\frac{\pi}{2}m+\Theta)\, 2\cos(\frac{\pi}{2}n+\Theta)\}$

If $m=n$:

$$r_{xx}[m,n] = E\{(X[n])^2\} = 2$$

If $m \neq n$: ↙Using the formula.

$$r_{xx}[m,n] = E\{4 \cdot \tfrac{1}{2}[\cos(\tfrac{\pi}{2}(m+n)+2\Theta) + \cos(\tfrac{\pi}{2}(m-n))]\}$$

$$= \underbrace{2E\{\cos(\tfrac{\pi}{2}(m+n)+2\Theta)\}}_{\text{Integrates to zero.}} + 2E\{\cos(\tfrac{\pi}{2}(m-n))\}$$

$$= 0 + 2\int_0^{2\pi} \cos(\tfrac{\pi}{2}(m-n))\, \frac{1}{2\pi}\, d\Theta$$

$$= 2\cos(\tfrac{\pi}{2}(m-n))$$

So,

$$r_{xx}[m,n] = 2\cos(\tfrac{\pi}{2}(m-n))$$

d) The mean is constant.

$r_{xx}[m,n]$ depends only on the difference $m-n$.

Therefore, $X[n]$ is wide-sense stationary.

2. (25 pts.) A wide-sense stationary, unit mean random process X[n] has autocorrelation

$$r_{XX}[n] = \begin{cases} 2, & n = 0 \\ 1.5, & n = \pm 1 \\ 1, & \text{else} \end{cases}$$

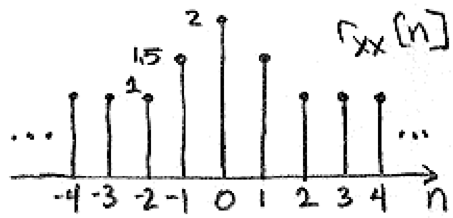
The process is passed through a first order discrete-time differentiator, described by the filter equation

$$Y[n] = X[n] - X[n-1]$$

a. (4) Find the mean E{Y[n]} of the output process

b. (20) Find the autocorrelation $r_{YY}[n]$ of the output process.

c. (1) How does the differentiator affect the "correlateness" of the input process?



a) X[n] has unit mean and is wide-sense stationary.

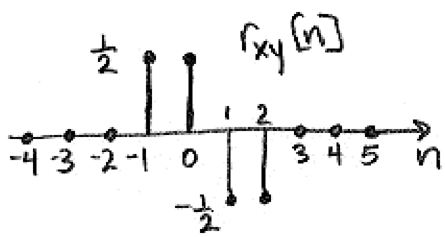
$h[n] = \delta[n] - \delta[n-1]$

$\bar{Y} = \sum_{n=-\infty}^{\infty} h[n]\,\overline{X[n]} = \bar{X} \sum_{n=-\infty}^{\infty} h[n] = 1 \cdot [1-1] = \underline{\underline{0}}$

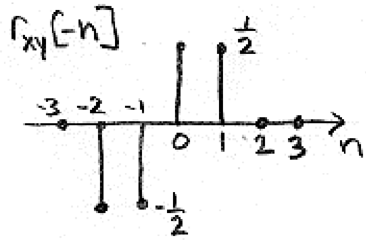
b) $r_{xy}[n] = h[n] * r_{xx}[n]$

$r_{yy}[n] = h[n] * r_{xy}[-n]$

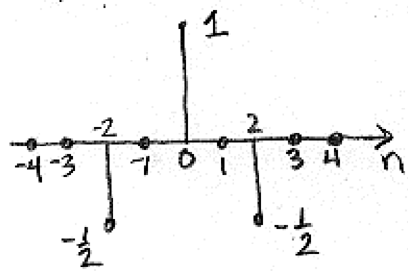
plugging $r_{xx}[n]$ into the difference equation gives $r_{xy}[n]$.



2. (continued)



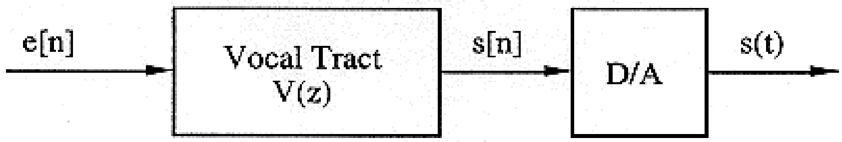
Plugging $r_{xy}[-n]$ into the difference equation gives $r_{yy}[n]$



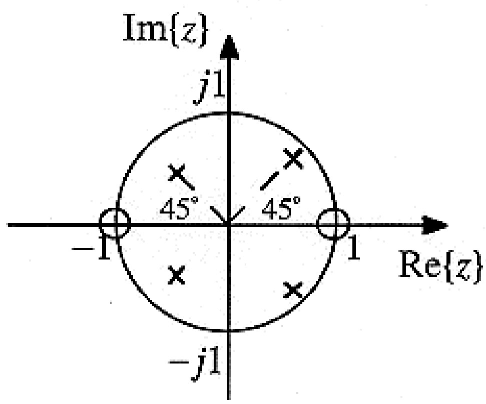
$$r_{yy}[n] = \begin{cases} 1, & n=0 \\ -0.5, & n=\pm 2 \\ 0, & \text{else} \end{cases}$$

c) The differentiator removes the "DC" component of the signal.

3. (25 pts.) The digital synthesizer for voiced speech shown below operates at a 10 kHz sampling rate.



The excitation is given by $e[n] = \sum_{k=-\infty}^{\infty} \delta[n - 200k]$. The vocal tract transfer function $V(z)$ has poles and zeros at the locations shown below:



a. (10) What is the pitch period in seconds?

b. (10) Find the formant frequencies of s(t) in analog units of Hz, and rank them according to their strength, *i.e.* how peaked the vocal tract response is at the corresponding frequency.

c. (5) Sketch roughly what the continuous-time Fourier transform (CTFT) of the waveform s(t) would look like.

a) e[n] gives one pulse every 200 samples. Samples are spaced $T_s = \frac{1}{f_s}$ apart.

$$P = \frac{200}{10{,}000} = \frac{1}{50} = 0.02 \text{ seconds}$$

b) There are two pole-conjugate pairs. One pair is at $\pm\frac{\pi}{4}$. The other is at $\pm\frac{3\pi}{4}$. The pair at $\pm\frac{\pi}{4}$ is closest to the unit circle and will give the most peaked response.

$$f = f_s \frac{\omega}{2\pi}$$

3. (continued)

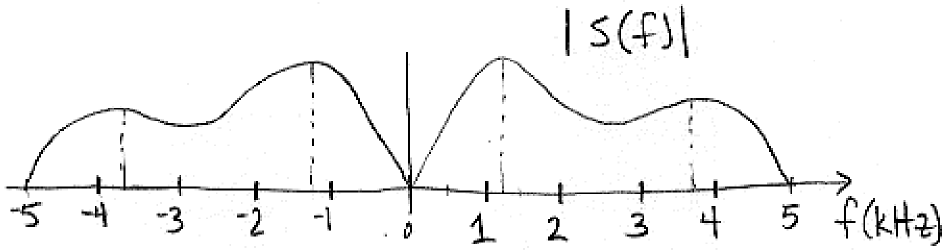
$$f_1 = 10\,\text{kHz}\left(\frac{\pi/4}{2\pi}\right) = 1250\ \text{Hz}$$

$$f_2 = 10\,\text{kHz}\left(\frac{3\pi/4}{2\pi}\right) = 3750\ \text{Hz}$$

f_1 is the strongest.

c)

Assuming ideal D/A including LPF



4. (25 pts) Consider the continuous-time (CT) signal x(t) below:

$$x(t) = [1 + \cos(2\pi 100 t)]\cos(2\pi 5000 t)$$

a. (8) Sketch x(t).

Consider x(t) as a model for a speech waveform

b. (1) Does it represent a voiced or unvoiced phoneme?

c. (3) What is the pitch period?

d. (3) How many formants are there, and at which frequencies do they occur?

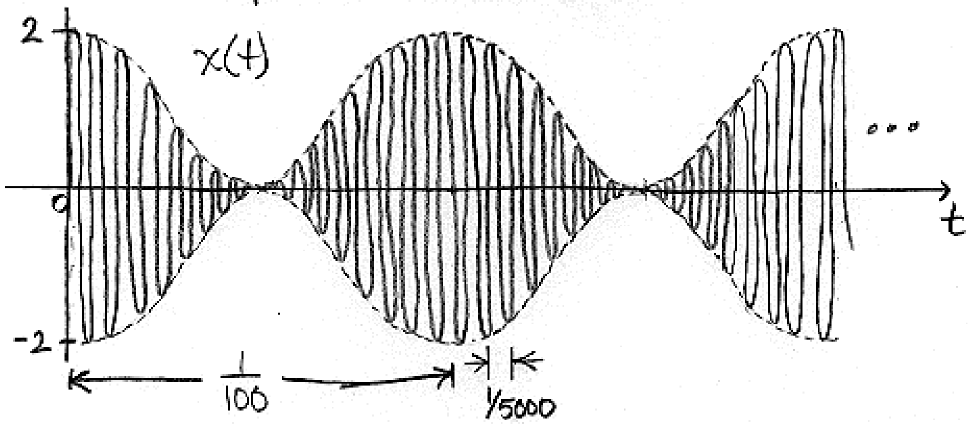
Define a CT version of the short time Fourier transform (STCTFT) as

$$X(f,t) = \int_{-\infty}^{\infty} x(s)w(t-s)e^{-j2\pi fs}\,ds$$

where w(t) is the window function that time-limits the Fourier transform.

e. (10) Assume that w(t) = rect(t/0.1). Find the STCTFT of the signal $x(t) = e^{j2\pi f_0 t}$.

a) This is an amplitude modulated cosine.



b) Voiced (periodic)

c) $P = \frac{1}{5000} = 0.2\text{ms}$

d) $x(t) = [1 + \cos(2\pi 100 t)]\cos(2\pi 5000 t)$

Using formula from problem 1.

$= \cos(2\pi 5000t) + \cos(2\pi 100 t)\cos(2\pi 5000 t)$

$= \cos(2\pi 5000 t) + \frac{1}{2}\cos(2\pi 4900 t) + \frac{1}{2}\cos(2\pi 5100 t)$

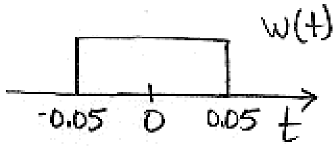
Three frequencies are present.

3 formants are present. They occur at:

4900 Hz, 5000 Hz, and 5100 Hz.

4. (continued)

e)



$$t-s=0.05 \Rightarrow s=t-0.05$$
$$t-s=-0.05 \Rightarrow s=t+0.05$$

$$X(f,t)=\int_{-\infty}^{\infty} e^{j2\pi f_0 s}\, w(t-s)\, e^{-j2\pi f s}\, ds$$

$$=\int_{t+0.05}^{t-0.05} e^{j2\pi f_0 s} e^{-j2\pi f s}\, ds = \int_{t+0.05}^{t-0.05} e^{-j2\pi(f-f_0)s}\, ds$$

$$=\frac{e^{-j2\pi(f-f_0)s}}{-j2\pi(f-f_0)}\Bigg|_{t+0.05}^{t-0.05} = \frac{1}{-j2\pi(f-f_0)}\left(e^{-j2\pi(f-f_0)(t-0.05)}-e^{-j2\pi(f-f_0)(t+0.05)}\right)$$

$$=e^{-j2\pi(f-f_0)t}\left(\frac{e^{j2\pi(f-f_0)0.05}-e^{-j2\pi(f-f_0)0.05}}{-j2\pi(f-f_0)}\right)$$

$$=e^{-j2\pi(f-f_0)t}\,\frac{\sin(2\pi(f-f_0)0.05)}{-\pi(f-f_0)}$$

$$=e^{-j2\pi(f-f_0)t}\,\frac{\sin(\pi(f-f_0)0.1)}{-\pi(f-f_0)\left(\frac{0.1}{0.1}\right)}$$

$$=-0.1\,\mathrm{sinc}(0.1(f-f_0))\, e^{-j2\pi(f-f_0)t}$$

Another approach is to use transform pairs ...

1.	_______
2.	_______
3.	_______
4.	_______
Total	_______

$$X(f,t) = \int_{-\infty}^{\infty} x(s)\, w(t-s)\, e^{-j2\pi f s}\, ds$$

One can look at this as the Fourier transform of a signal multiplied by a time shifted window. Thus, $X(f,t)$ is the convolution of the transform of $x(s)$ with the transform of $w(t-s)$.

$$x(s) = e^{j2\pi f_0 s} \overset{CTFT}{\longleftrightarrow} \delta(f-f_0) = X(f)$$

$$w(t-s) = \text{rect}\left((t-s)/0.1\right) \overset{CTFT}{\longleftrightarrow} |-0.1|\,\text{sinc}(-0.1 f)\, e^{-j2\pi f t} = W(f)$$

$$= \text{rect}\left(\tfrac{-1}{0.1}(s-t)\right) \qquad = -0.1\,\text{sinc}(0.1 f)\, e^{-j2\pi f t}$$

so $X(f,t) = X(f) * W(f)$

$$\underline{\underline{= -0.1\,\text{sinc}(0.1(f-f_0))\, e^{-j2\pi(f-f_0)t}}}$$