October 30, 1998 **Name:** SOLUTIONS

EE 438 **Exam No. 2** **Fall 1998**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the causal LTI system described by the difference equation

$$y[n] = x[n] + y[n-1]$$

   a. (14) Use the ZT to find the impulse response h[n] for this system.
   b. (10) Use the ZT to find the response y[n] of the system to the input x[n] = u[n].
   c. (1) Is this system BIBO stable?

a) $y[n] = x[n] + y[n-1]$

$$Y(z) = X(z) + z^{-1} Y(z)$$

$$Y(z)(1 - z^{-1}) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}}$$

The system has a pole at $z = 1$. The system is causal, so choose the Region of Convergence as $|z| > 1$.

$$h[n] \overset{ZT}{\longleftrightarrow} H(z)$$

$$h[n] = 1^n u[n] = u[n].$$

Note: The impulse response has infinite duration.

1. (continued)

b) $u[n] \overset{ZT}{\longleftrightarrow} \dfrac{1}{1-z^{-1}}, \quad |z|>1$

$$Y(z) = X(z)H(z) = \frac{1}{1-z^{-1}} \cdot \frac{1}{1-z^{-1}} = \frac{1}{(1-z^{-1})^2}, \quad |z|>1$$

$$n\,a^n u[n] \overset{ZT}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z|>|a|$$

$$(n+1)\,a^{(n+1)} u[n+1] \overset{ZT}{\longleftrightarrow} \frac{a}{(1-az^{-1})^2} \quad |z|>|a|$$

$$y[n] = (n+1)\,1^{n+1} u[n+1] = (n+1)\,u[n+1] \longrightarrow = 0 \text{ at } n=-1$$

$$= (n+1)\,u[n]$$

c) There is a pole on the unit circle. Therefore, the ROC does not contain the unit circle, so the system is not BIBO stable.

To see this, consider the response to the bounded input $x[n] = u[n]$. As $n \to \infty$, $y[n] \to \infty$.

2. (25 pts.) Let x[n], n = 0,…,N-1 be an N point signal with N point DFT $X^N[k],\ k=0,\dots,N-1$. Now generate a new 2N point signal y[n], n = 0,…, 2N-1 by simply repeating x[n], i.e.

$$y[n]=\begin{cases} x[n], & 0\le n\le N-1\\ x[n-N], & N\le n\le 2N-1\end{cases}$$

Find a simple expression for the 2N point DFT $Y^{2N}[k],\ k=0,\dots,2N-1$ in terms of $X^N[k],\ k=0,\dots,N-1$.

$$Y[k]=\sum_{n=0}^{2N-1} y[n]\,e^{-j2\pi kn/(2N)}$$

$$=\sum_{n=0}^{N-1} x[n]\,e^{-j2\pi kn/(2N)} + \sum_{m=N}^{2N-1} x[m-N]\,e^{-j2\pi km/(2N)}$$

We want ↗ to be in the form of a regular DFT. So do a change of variables. Let $n=m-N$.

Then: $m=n+N$

when $m=N$, $n=N-N=0$

when $m=2N-1$, $n=2N-1-N$
$=N-1$

So, $$Y[k]=\sum_{n=0}^{N-1} x[n]\,e^{-j2\pi kn/(2N)} + \sum_{n=0}^{N-1} x[n]\,e^{-j2\pi k(n+N)/2N}$$

$$=\sum_{n=0}^{N-1} x[n]\,e^{-j2\pi(\frac{k}{2})n/N} + e^{-j2\pi kN/(2N)}\sum_{n=0}^{N-1} x[n]\,e^{-j2\pi(\frac{k}{2})n/N}$$

$$=X[\tfrac{k}{2}]+e^{-j\pi k}X[\tfrac{k}{2}]$$

$$=X[\tfrac{k}{2}](1+(-1)^k)$$

$$=\begin{cases}2X[\frac{k}{2}] & k \text{ even}\\ 0 & k \text{ odd}\end{cases}$$

Alternate Solution

2. (25 pts.) Let x[n], n = 0,…,N-1 be an N point signal with N point DFT $X^N[k],\ k=0,\ldots,N-1$. Now generate a new 2N point signal y[n], n = 0,…, 2N-1 by simply repeating x[n], i.e.

$$y[n]=\begin{cases} x[n], & 0\le n\le N-1 \\ x[n-N], & N\le n\le 2N-1 \end{cases}$$

Find a simple expression for the 2N point DFT $Y^{2N}[k],\ k=0,\ldots,2N-1$ in terms of $X^N[k],\ k=0,\ldots,N-1$.

$$Y^{2N}[k] = \sum_{n=0}^{2N-1} y[n]\, e^{-j2\pi kn/2N}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/2N} + \sum_{n=N}^{2N-1} x[n-N]\, e^{-j2\pi kn/2N}$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{l=0}^{N-1} X^N[l]\, e^{j2\pi ln/N}\, e^{-j2\pi kn/2N}$$

$$= \sum_{n=N}^{2N-1} \frac{1}{N} \sum_{l=0}^{N-1} X^N[l]\, e^{j2\pi l(n-N)/N}\, e^{-j2\pi kn/2N}$$

but $e^{j2\pi l(n-N)/N} = e^{j2\pi ln/N}$ so two summands are same and can be combined as one sum:

$$Y^{2N}[k] = \sum_{n=0}^{2N-1} \frac{1}{N} \sum_{l=0}^{N-1} X^N[l]\, e^{j2\pi ln/N}\, e^{-j2\pi kn/2N}$$

Interchange order of summation & examine

$$\frac{1}{N}\sum_{n=0}^{2N-1} e^{j2\pi n(2l-k)/2N} = \frac{1}{N}\,\frac{1-e^{j2\pi(2l-k)}}{1-e^{j2\pi(2l-k)/2N}}$$

$$= \begin{cases} 2, & 2l = k \mod 2N \\ 0, & \text{else} \end{cases}$$

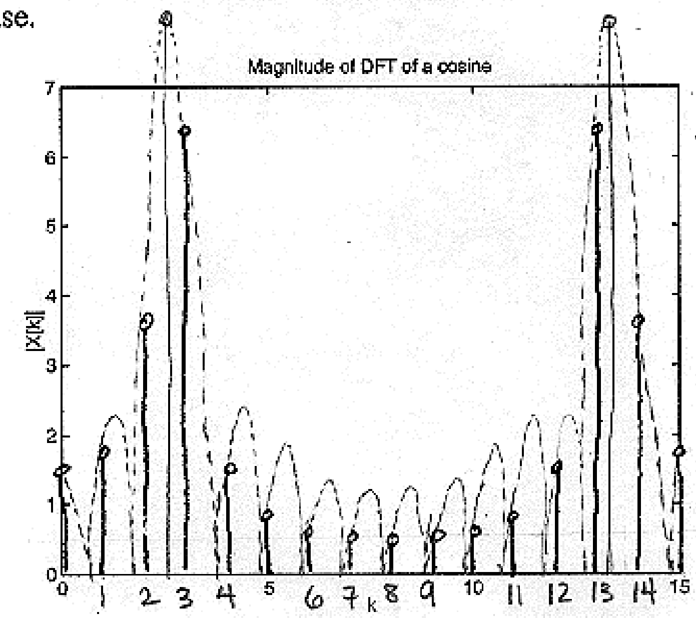
2. (continued)

Now consider separately k even & k odd.
When k is even, $2\ell = k$ is satisfied when $\ell = k/2$. When k is odd, $2\ell = k$ is never satisfied for any ℓ

Thus

$$Y^{2N}[k] = \begin{cases} 2X^N[k/2], & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

3. (25 pts.) The plot below shows the magnitude of the DFT of a cosine x[n] which was obtained by sampling an analog waveform at a 10kHz rate.
   a. (10) Estimate the digital frequency ω_d in radians/sample of this signal.
   b. (5) Assuming that the analog waveform was not undersampled, what is the corresponding analog frequency f_a?
   c. (5) Find the digital frequency ω_d' closest to ω_d for which there is no leakage or picket fence effect.
   d. (5) Sketch approximately what the magnitude of the DFT would look like in this case.



We are seeing a sampled version of two delta functions convolved with a periodic sinc function.

The spectrum of a cosine consists of two delta functions. In this case, the delta functions are not located at integer values of k, and we see leakage. The first delta is between k=2 and k=3, and is closer to 3. The second delta is between k=13 and k=14, and is closer to 13.

The DFT of a cosine has the form

$$\cos(2\pi k_0 n/N),\ 0 \le n \le N-1 \xleftrightarrow{DFT} \frac{N}{2}\{\delta[k-k_0]+\delta[k-(N-k_0)]\},$$
$$0 \le k \le N-1$$

If we choose $k_0 = 2.75$ then

$$\omega_d = 2\pi\left(\frac{k_0}{N}\right) = \frac{2\pi\ 2.75}{16} \approx 0.344\pi = 1.080 \text{ radians/sample.}$$

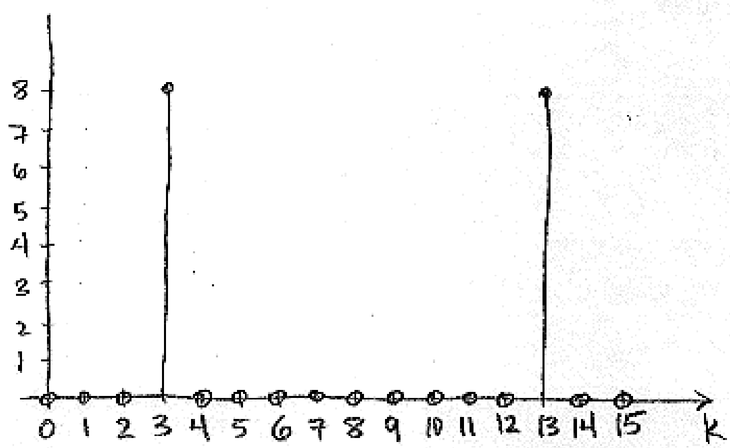
3. (continued)

b) $f_a = \frac{k_0}{N} f_s = \frac{\omega_d}{2\pi} f_s = \frac{2.75}{16} 10{,}000 \approx 1719 \text{ Hz}$

c) If k_0 is an integer, we sample at the zero-crossings of the periodic sinc. The closest integer to k_0 is 3.

$$\omega_d' = \frac{3}{16} 2\pi = \frac{3\pi}{8}$$

d)



4. (25 pts.) Derive the decimation in time fast Fourier transform algorithm for a 6 point DFT. Show a complete flow diagram, including all twiddle factors for your FFT.

There are two ways to factor the 6-point DFT.

Way #1

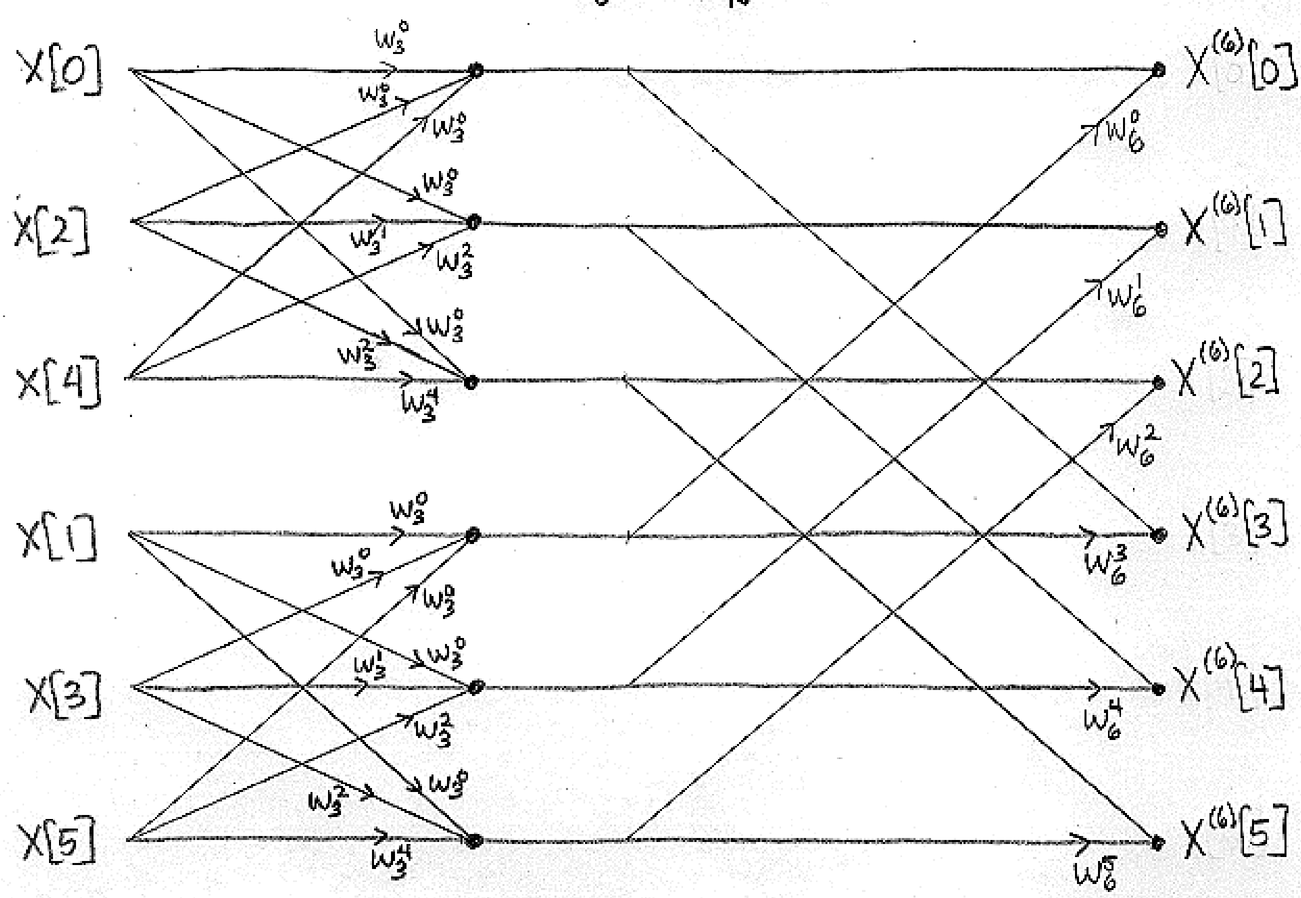
$$X^{(6)}[k] = \sum_{n=0}^{5} x[n] e^{-j2\pi kn/6}$$

$$= \sum_{n=0}^{2} x[2n] e^{-j2\pi k \cdot 2n/6} + \sum_{n=0}^{2} x[2n+1] e^{-j2\pi k(2n+1)/6}$$

$$= \sum_{n=0}^{2} x[2n] e^{-j2\pi kn/3} + e^{-j2\pi k/6} \sum_{n=0}^{2} x[2n+1] e^{-j2\pi kn/3}$$

$$= \underbrace{\sum_{n=0}^{2} x[2n] W_3^{nk}}_{\text{3-point DFT of } x[2n]} + \underbrace{W_6^k}_{\text{twiddle factor}} \underbrace{\sum_{n=0}^{2} x[2n+1] W_3^{nk}}_{\text{3-point DFT of } x[2n+1]}, \text{ where } W_N = e^{-j2\pi/N}$$

Note: $W_N^0 = 1$



4. (continued)

Way #2

$$X^{(6)}[k] = \sum_{n=0}^{5} x[n] e^{-j2\pi kn/6}$$

$$= \sum_{n=0}^{1} x[3n] e^{-j2\pi k3n/6} + \sum_{n=0}^{1} x[3n+1] e^{-j2\pi k(3n+1)/6} + \sum_{n=0}^{1} x[3n+2] e^{-j2\pi k(3n+2)/6}$$

$$= \sum_{n=0}^{1} x[3n] e^{-j2\pi kn/2} + e^{-j2\pi k/6} \sum_{n=0}^{1} x[3n+1] e^{-j2\pi kn/2} + e^{-j2\pi 2k/6} \sum_{n=0}^{1} x[3n+2] e^{-j2\pi kn/2}$$

$$= \underbrace{\sum_{n=0}^{1} x[3n] W_2^{nk}}_{\text{2-point DFT}} + W_6^{k} \underbrace{\sum_{n=0}^{1} x[3n+1] W_2^{nk}}_{\text{2-point DFT}} + W_6^{2k} \underbrace{\sum_{n=0}^{1} x[3n+2] W_2^{nk}}_{\text{2-point DFT}}$$

Twiddle Factors