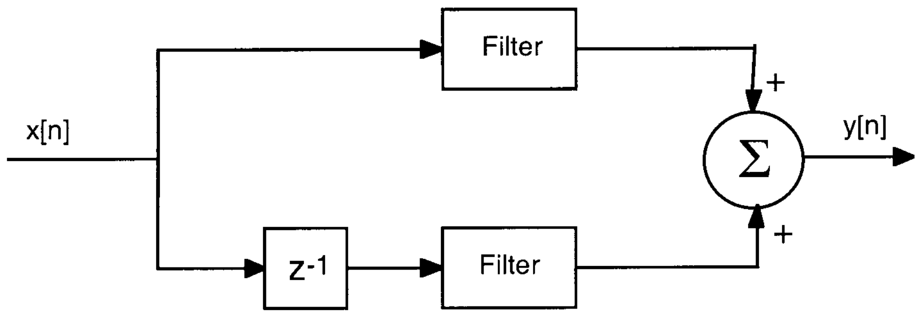
September 28, 1998 **Name:** Solutions

EE 438 **Exam No. 1** **Fall 1998**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the system shown below



where z^{-1} denotes a unit sample delay; and the filter is defined by the following difference equation:

$$y[n] = 0.5(x[n] + x[n-1])$$

a. (10) Find the frequency response $H(\omega)$ for the system in terms of the frequency response $F(\omega)$ of the filter

b. (5) Find a simple expression for the frequency response $F(\omega)$ of the filter. You don't need to separate it into magnitude and phase.

c. (10) Determine simple expressions for the magnitude and phase of the frequency response $H(\omega)$ of the overall system.

a) $Y(\omega) = X(\omega)F(\omega) + X(\omega)e^{-j\omega}F(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = F(\omega)(1+e^{-j\omega}) = F(\omega)\,e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)$$

$$= F(\omega)\,2e^{-j\omega/2}\cos\left(\frac{\omega}{2}\right)$$

1. (continued)

b) Let $x[n] = e^{j\omega n} \Rightarrow y[n] = F(\omega)e^{j\omega n}$

$$F(\omega)\cancel{e^{j\omega n}} = 0.5(e^{j\omega n} + e^{j\omega(n-1)})$$

$$F(\omega) = 0.5(1+e^{-j\omega}) = \frac{1}{2}e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}})$$

$$= \underline{\underline{e^{-j\frac{\omega}{2}}\cos\left(\frac{\omega}{2}\right)}}$$

c) $H(\omega) = e^{-j\frac{\omega}{2}}\cos\left(\frac{\omega}{2}\right) 2e^{-j\frac{\omega}{2}}\cos\left(\frac{\omega}{2}\right)$

$$= 2e^{-j\omega}\cos^2\left(\frac{\omega}{2}\right)$$

$$\underline{\underline{|H(\omega)| = 2\cos^2\left(\frac{\omega}{2}\right)}}$$

$$\angle H(\omega) = \angle 2 + \angle e^{-j\omega} + \angle \cos^2\left(\frac{\omega}{2}\right)$$ ← Always real and positive.

$$= 0 + (-\omega) + 0$$

$$\underline{\underline{\angle H(\omega) = -\omega}}$$

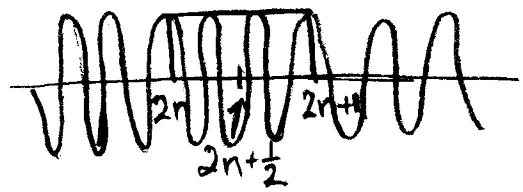
2. (25 pts.) Consider the continuous-time signal

$$x(t) = \begin{cases} \cos[2\pi(100)t], & 2n < t \le 2n+1, \text{ for some integer } n \\ 0, & \text{else} \end{cases}$$

a. (18) Use standard functions and CTFT relations to find the CTFT $X(f)$. (Do *not* directly evaluate any integrals!)

b. (7) Sketch $|X(f)|$.

a)



$$x(t) = \cos(2\pi 100 t)\,\text{rect}\left(t - \left(2n + \tfrac{1}{2}\right)\right)$$

$$X(f) = \frac{1}{2}\left[\delta(f-100) + \delta(f+100)\right] * \text{sinc}(f)\,e^{-j2\pi f(2n+\frac{1}{2})}$$

$$= \frac{1}{2}\left[\text{sinc}(f-100)\,e^{-j2\pi(f-100)(2n+\frac{1}{2})} + \text{sinc}(f+100)\,e^{-j2\pi(f+100)(2n+\frac{1}{2})}\right]$$

Optional simplification:

$$e^{-j2\pi(f-100)(2n+\frac{1}{2})} = e^{-j2\pi(2nf + \frac{1}{2}f \underbrace{- 200n - 50}_{\text{these terms give integer multiples of } 2\pi})} = e^{-j2\pi(2nf + \frac{1}{2}f)} = e^{-j\pi f(4n+1)}$$

$$e^{-j2\pi(f+100)(2n+\frac{1}{2})} = e^{-j2\pi(2nf + \frac{1}{2}f)} = e^{-j\pi f(4n+1)}$$

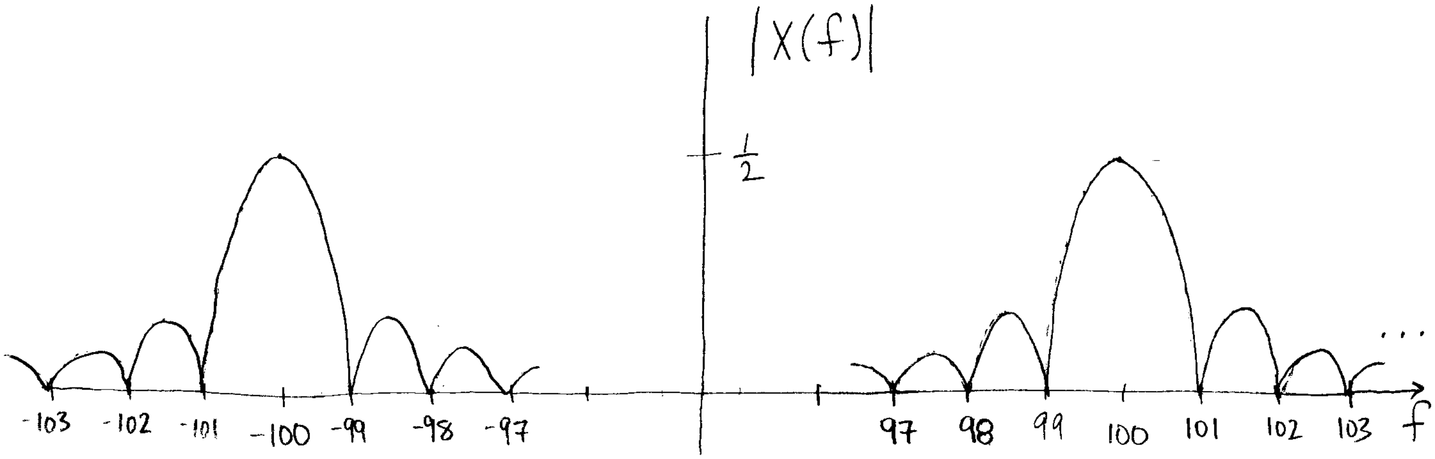
$$X(f) = \frac{1}{2}\left[\text{sinc}(f-100)\,e^{-j\pi f(4n+1)} + \text{sinc}(f+100)\,e^{-j\pi f(4n+1)}\right]$$

$$= \frac{1}{2}\left[\text{sinc}(f-100) + \text{sinc}(f+100)\right]e^{-j\pi f(4n+1)}$$

2. (continued)

b) $|X(f)| = \left|\frac{1}{2}\left(\text{sinc}(f-100) + \text{sinc}(f+100)\right)\right| \cdot \underbrace{\left|e^{-j\pi f(4n+1)}\right|}_{\to 1}$

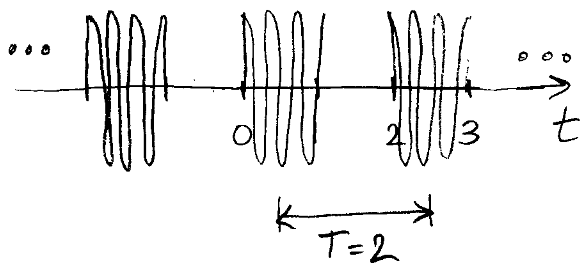
$= \left|\frac{1}{2}\left(\text{sinc}(f-100) + \text{sinc}(f+100)\right)\right|$



(4b)

Alternate way of looking at the problem:

$$x(t) = \begin{cases} \cos[2\pi(100)t] & 2n < t \le 2n+1 \quad \text{for } n = \cdots -2, -1, 0, 1, 2, \cdots \\ 0 & \text{else.} \end{cases}$$



(a)

$$x(t) = \cos(2\pi 100 t)\ \mathrm{rep}_2\left(\mathrm{rect}\left(t-\tfrac{1}{2}\right)\right)$$

$$X(f) = \frac{1}{2}\left[\delta(f-100) + \delta(f+100)\right] * \frac{1}{2}\,\mathrm{comb}_{\frac{1}{2}}\left(\mathrm{sinc}(f)\, e^{-j2\pi f \frac{1}{2}}\right)$$

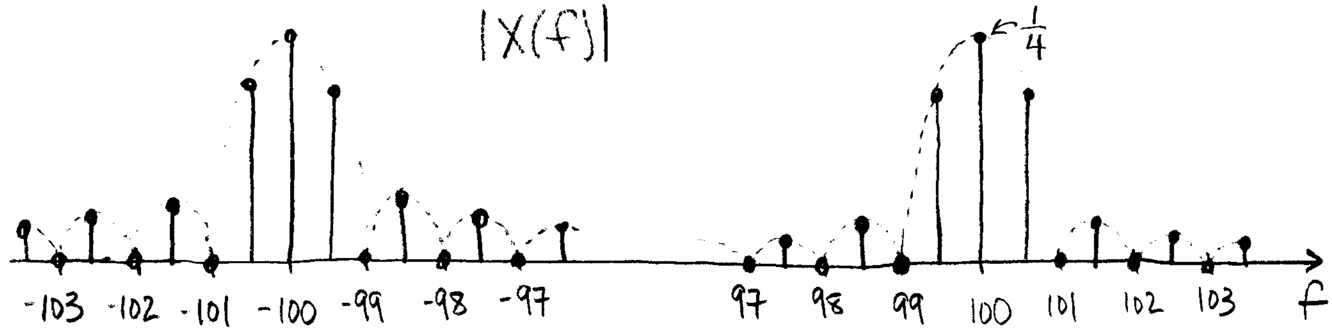
$$= \frac{1}{2}\left[\delta(f-100) + \delta(f+100)\right] * \frac{1}{2}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(k\tfrac{1}{2}\right) e^{-j\pi\left(k\frac{1}{2}\right)}\, \delta\left(f - k\tfrac{1}{2}\right)$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \left(\delta(l-100) + \delta(l+100)\right) \sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(k\tfrac{1}{2}\right) e^{-j\pi k \frac{1}{2}}\, \delta\left(f - l - k\tfrac{1}{2}\right) dl$$

$$= \frac{1}{4}\sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(k\tfrac{1}{2}\right) e^{-j\pi k \frac{1}{2}} \left(\delta\left(f-100-k\tfrac{1}{2}\right) + \delta\left(f+100-k\tfrac{1}{2}\right)\right)$$

(b)

$$|X(f)| = \frac{1}{4}\sum_{k=-\infty}^{\infty} \left|\mathrm{sinc}\left(k\tfrac{1}{2}\right)\right| \left(\delta\left(f-100-k\tfrac{1}{2}\right) + \delta\left(f+100-k\tfrac{1}{2}\right)\right)$$



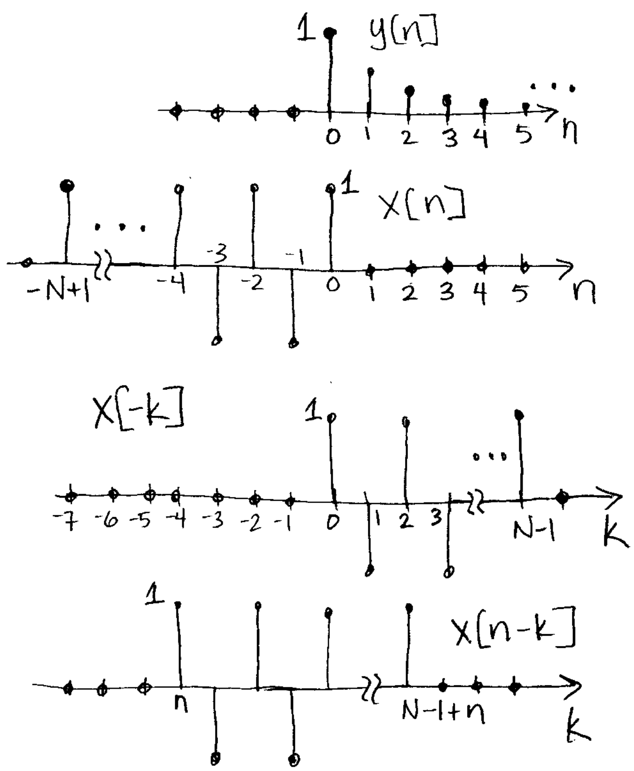
3. (25 pts.). Find the convolution $z[n] = x[n] * y[n]$ of the two signals shown below.

$$x[n] = \begin{cases} (-1)^n, & -N+1 \le n \le 0 \\ 0, & \text{else}; \end{cases}$$

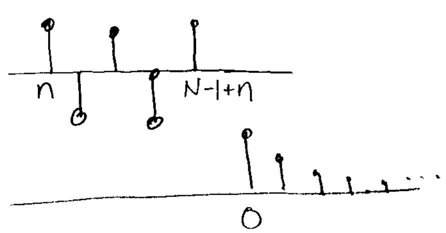
$$y[n] = \begin{cases} 2^{-n}, & 0 \le n \\ 0, & \text{else}; \end{cases}$$

where N is a positive integer that is greater than 1. Your answer should be valid for any such N.

$$z[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k] = \sum_{k=-\infty}^{\infty} y[n]\, x[n-k]$$



Case 1: No overlap



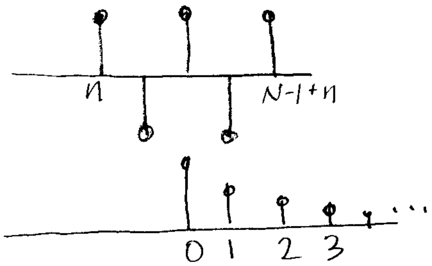
Occurs when $N-1+n < 0$

$\Rightarrow n < -N+1$

$z[n] = 0$ for $n < -N+1$

3. (continued)

Case 2: Partial Overlap



Occurs when $N-1+n \geq 0$ and $n<0$

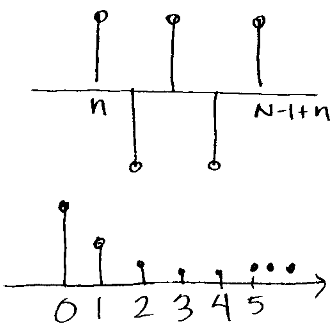
$\Rightarrow n \geq -N+1$ and $n<0$

$$z[n] = \sum_{k=0}^{N-1+n} y[n]\,x[n-k]$$

$$z[n] = \sum_{k=0}^{N-1+n} 2^{-k}(-1)^{n-k} = \sum_{k=0}^{N-1+n} (-2)^{-k}(-1)^{n} = (-1)^n \sum_{k=0}^{N-1+n} \left(-\tfrac{1}{2}\right)^k$$

$$= (-1)^n \left(\frac{1-\left(-\frac{1}{2}\right)^{N+n}}{1-\left(-\frac{1}{2}\right)} \right) = \frac{2}{3}(-1)^n\left(1-\left(-\tfrac{1}{2}\right)^{N+n}\right)$$

Case 3: $x[n-k]$ completely overlaps $y[k]$



Occurs when $n \geq 0$

$$z[n] = \sum_{k=n}^{N-1+n} 2^{-k}(-1)^{n-k}$$

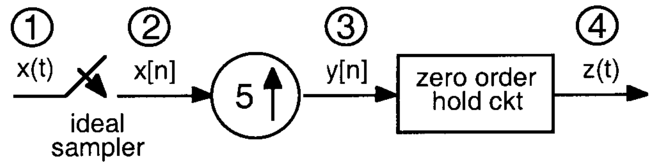
Let $m = k-n$, then $k = m+n$.
when $k=n$, $m = n-n = 0$
$k = N-1+n$, $m = N-1$

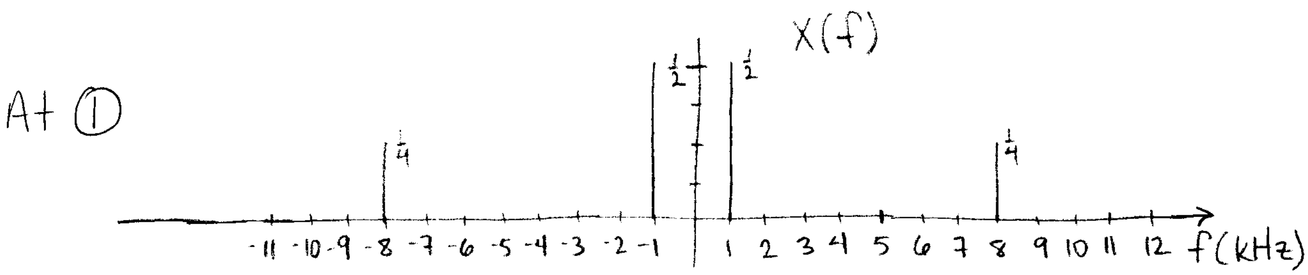
$$= \sum_{m=0}^{N-1} 2^{-(m+n)}(-1)^{n-(m+n)}$$

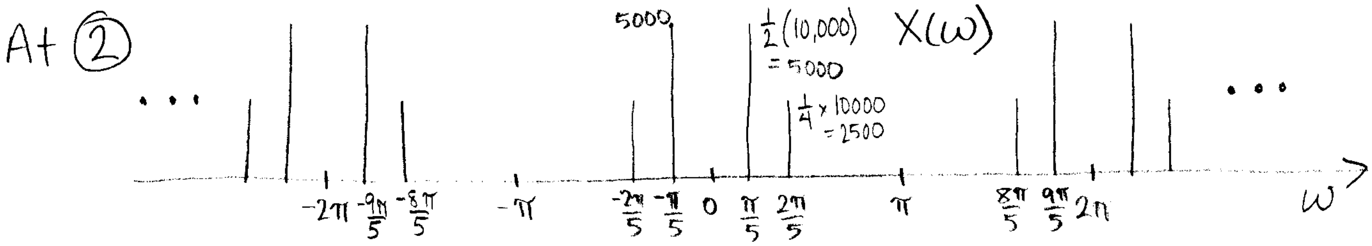
$$z[n] = 2^{-n} \sum_{m=0}^{N-1} \left(-\tfrac{1}{2}\right)^m = 2^{-n}\,\frac{1-\left(-\frac{1}{2}\right)^N}{1-\left(-\frac{1}{2}\right)} = \frac{2^{-n+1}}{3}\left(1-\left(-\tfrac{1}{2}\right)^N\right)$$

$$\boxed{\text{So, } z[n] = \begin{cases} 0 & n < -N+1 \\ \frac{2}{3}(-1)^n\left(1-\left(-\frac{1}{2}\right)^{N+n}\right) & -N+1 \leq n < 0 \\ \frac{2^{-n+1}}{3}\left(1-\left(-\frac{1}{2}\right)^N\right) & 0 \leq n \end{cases}}$$

4. (25 pts.) Consider the signal $x(t) = \cos(2\pi(1000)t) + 0.5\cos(2\pi(8000)t)$, which is processed by the system shown below. The signal is sampled with an ideal sampler operating at a 10 kHz rate. It is then upsampled by a factor of 5, and finally reconstructed using a zero-order-hold circuit operating at a frequency of 50 kHz. Sketch the *spectra* of the signals at the circled and numbered points 1 through 4. Be sure to label all dimensions in your plots.







The spectrum is periodic in 2π.

The 1kHz signal is at $\omega = 2\pi \frac{1\,\text{kHz}}{10\,\text{kHz}} = \frac{1}{5}\pi$

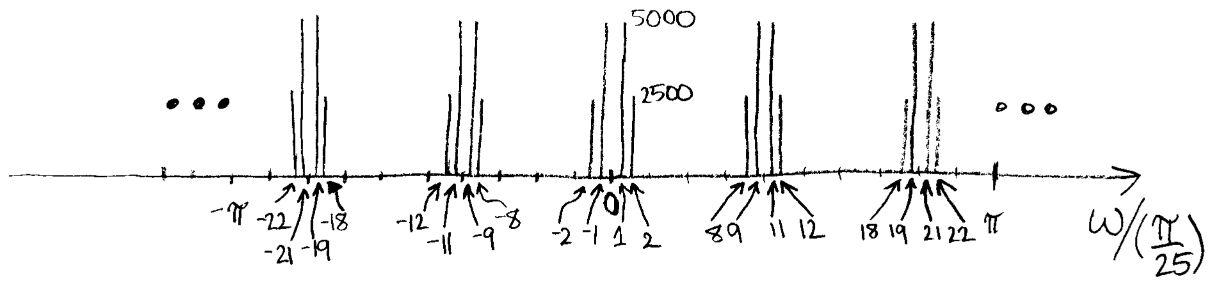
The amplitudes are multiplied by the sample frequency.

The 8kHz signal is aliased to $\pi - \left(2\pi \frac{8\,\text{kHz}}{10\,\text{kHz}} - \pi\right) = \frac{2}{5}\pi$

However, we're mainly interested in the relative amplitudes.

4. (continued)

At ③ $Y(\omega) = X(5\omega)$



At ④ For a zero-order hold circuit, we convolve our samples with a shifted rect function $\mathrm{rect}\left(\frac{t - T_s/2}{T_s}\right)$.

Then the spectrum becomes

$$Z(f) = \left(X(5\omega)\Big|_{f=\frac{\omega}{2\pi}f_s}\right)|T_s|\,\mathrm{sinc}(T_s f)\,e^{-j2\pi f T_s/2}$$

$$= \left(X(5\omega)\Big|_{f=\frac{\omega}{2\pi}f_s}\right)\left|\frac{1}{f_s}\right|\,\mathrm{sinc}\left(\frac{f}{f_s}\right)e^{-j\pi f/f_s}$$

$|Z(f)|$