28 November 2018 **Name:** Qingyu Yang - prob. 1

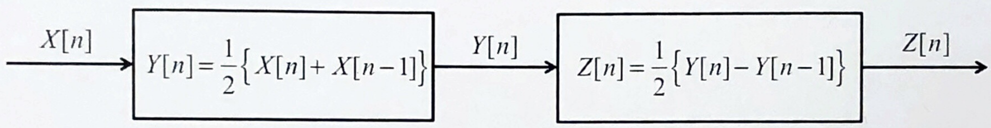
ECE 438 **Exam No. 3** **Fall 2018**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Let $X[n]$ be a wide-sense stationary sequence of random variables with zero mean and autocorrelation function

$$r_{XX}[n] = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & |n| = 1 \\ 0, & \text{else} \end{cases}$$

Suppose that this sequence is processed with the following cascade of two filters to generate the output sequence $Z[n]$:



a. (7) Find the mean of the sequence $Z[n]$.

b. (18) Find the cross-correlation $r_{XZ}[n]$ between X and Z.

a. $E[Z[n]] = E\left[\frac{1}{2}\{Y[n] - Y[n-1]\}\right]$

$= E\left[\frac{1}{2}\left\{\frac{1}{2}\{X[n] + X[n-1]\} - \frac{1}{2}\{X[n-1] + X[n-2]\}\right\}\right]$

$= \frac{1}{2} E\left[\frac{1}{2}X[n] + \frac{1}{2}X[n-1] - \frac{1}{2}X[n-1] - \frac{1}{2}X[n-2]\right]$

$= \frac{1}{4} E\left[X[n] - X[n-2]\right]$ $\boxed{3}$

$X[n]$ is wide-sense stationary, $E[X[n]] = 0$, $E[X[n-2]] = 0$ $\boxed{2}$

$E[Z[n]] = 0$ $\boxed{2}$

b. $r_{XZ}[n] = h[n] * r_{XX}[n]$ $\boxed{3}$, $Z[n] = \frac{1}{4}X[n] - \frac{1}{4}X[n-2]$

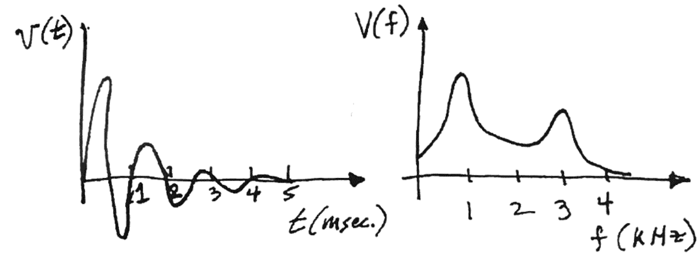
$h[n] = \frac{1}{4}\delta[n] - \frac{1}{4}\delta[n-2]$ $\boxed{3}$, $r_{XX}[n] = \frac{1}{2}\delta[n-1] + \delta[n] + \frac{1}{2}\delta[n+1]$ $\boxed{2}$

$r_{XZ}[n] = h[n] * r_{XX}[n] = \left\{\underset{\underset{n=0}{\uparrow}}{\frac{1}{4}}, 0, -\frac{1}{4}\right\} * \left\{\frac{1}{2}, \underset{\underset{n=0}{\uparrow}}{1}, \frac{1}{2}\right\} = \left\{\frac{1}{8}, \underset{\underset{n=0}{\uparrow}}{\frac{1}{4}}, 0, -\frac{1}{4}, -\frac{1}{8}\right\}$ $\boxed{5}$ $\boxed{5}$

or

$\frac{1}{4}r_{XX}[n] - \frac{1}{4}r_{XX}[n-2] = \left\{\frac{1}{8}, \underset{\underset{n=0}{\uparrow}}{\frac{1}{4}}, 0, -\frac{1}{4}, -\frac{1}{8}\right\}$

2. (25) Consider a voiced phoneme for which the time-domain, continuous-time vocal tract response $v(t)$ and corresponding vocal tract frequency response (CTFT) $V(f)$ are given below.



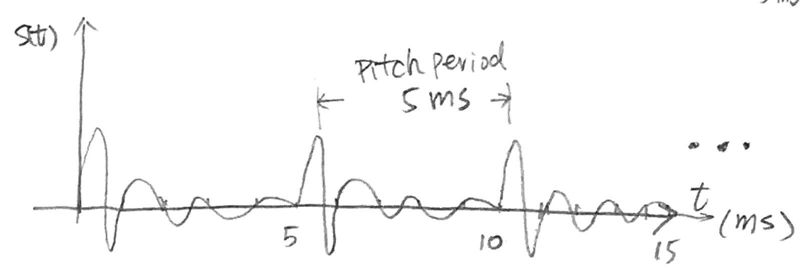
a. (8) Assume that the pitch frequency for the speaker is 200 Hz. Sketch what the continuous-time domain speech waveform $s(t)$ would look like in this case. Be sure to dimension all important quantities in the speech waveform.

b. (8) For your speech waveform $s(t)$ in part (a), sketch what the CTFT $S(f)$ would look like. Be sure to dimension all important quantities in $S(f)$.

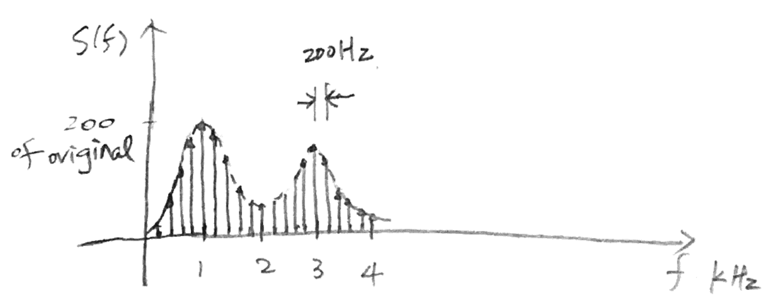
c. (9) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length of 500 samples. Carefully sketch the resulting spectrogram as a function of the discrete-time index n and digital frequency ω (radians/sample). Be sure to dimension all important quantities in your plot. Is this a wide-band or a narrow-band spectrogram?

a) Pitch period $= \frac{1}{200\text{ Hz}} = 5\text{ ms}$

$s(t) = v(t) * e(t)$, $e(t) = \text{rep}_{5ms}[\delta(t)]$



b) $S(f) = V(f) \cdot E(f)$, $E(f) = 200\ \text{comb}_{200Hz}\ 1$



2. (continued - 1)

c) From the given $V(F)$, we can read the format frequencies are

$F_1 = 1\text{ kHz}, \quad F_2 = 3\text{ kHz}.$

Pitch period in samples is $5\text{ ms} \cdot 10\text{ KHz} = 50$

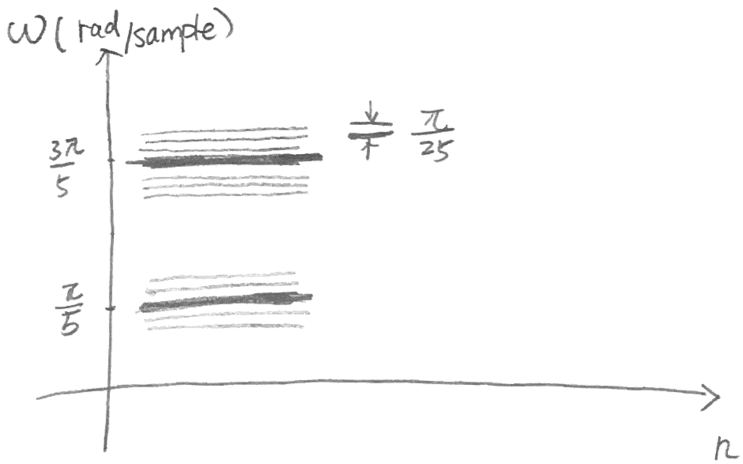
Here the STDTFT window length is 500 samples >> 50 (pitch period)

so this is $\boxed{\text{narrow band}}$ spectrogram.

$$\omega = F \cdot \frac{2\pi}{f_s}$$

$$\therefore \omega_1 = F_1 \frac{2\pi}{f_s} = 1\text{ kHz} \cdot \frac{2\pi}{10\text{ kHz}} = \frac{\pi}{5}, \quad \omega_2 = F_2 \frac{2\pi}{f_s} = \frac{3}{5}\pi$$

Pitch period in frequency: $200\text{ Hz} \cdot \frac{2\pi}{10\text{kHz}} = \frac{\pi}{25}$



This is Narrow band spectrogram.
The long time window length gives good frequency domain resolution and poor time domain resolution.

3. (25) Consider the signal

$$x[n]=\begin{cases}\cos(\pi n/4), & |n|\le 40\\ 0, & \text{else}\end{cases},$$

Assume a rectangular window

$$w[n]=\begin{cases}1, & |n|\le 20\\ 0, & \text{else}\end{cases}.$$

a. (17) Compute the STDTFT as defined below

$$\tilde{X}(\omega,n)=\sum_k x[k]w[n-k]e^{-j\omega k}$$

for the following cases. Be sure to express your answer in terms of the function $\text{psinc}_N(\omega)=\dfrac{\sin(\omega N/2)}{\sin(\omega/2)}$ for appropriate values of N:

i. $n=0$

ii. $n=40$

iii. $n=61$

b. (8) Sketch $|\tilde{X}(\omega,n)|$ for **all** n. Bes sure to dimension all important quantities.

a) $n=0$ $\quad \tilde{X}(\omega,0)=\sum_k x[k]w[-k]e^{-j\omega k} = \sum_k x[k]w[k]e^{-j\omega k}$

Product $\quad x[n]\cdot y[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega-\mu)Y(\mu)\,d\mu$

$x[n] = \cos(\frac{\pi}{4}n) \overset{DTFT}{\longleftrightarrow} \pi\, \text{rep}_{2\pi}\{\delta(\omega-\frac{\pi}{4})+\delta(\omega+\frac{\pi}{4})\}$

$w'[n]=\begin{cases}1 & n=0,1,\ldots 41-1\\ 0 & \text{else}\end{cases} \overset{DTFT}{\longleftrightarrow} \text{psinc}_{41}(\omega)\,e^{-j\omega(41-1)/2}$

$w[n]=w'[n+20] \overset{DTFT}{\longleftrightarrow} \text{psinc}_{41}(\omega)\,e^{-j\omega(20)}\cdot e^{-j\omega(-20)}$

$= \text{psinc}_{41}(\omega)$

$\tilde{X}(\omega,0) = \text{DTFT}\{x[k]\cdot w[k]\}$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi\left(\delta(\omega-\frac{\pi}{4}-\mu)+\delta(\omega+\frac{\pi}{4}-\mu)\right)\text{psinc}_{41}(\mu)\,d\mu$

6 pts

$= \frac{1}{2}\text{psinc}_{41}(\omega-\frac{\pi}{4}) + \frac{1}{2}\text{psinc}(\omega+\frac{\pi}{4})$

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3. (continued - 1)

ii $\boxed{n = 40}$

$$\tilde{X}(\omega, 40) = \sum_{k} x[k]\, w[40-k]\, e^{-j\omega k}$$

$$w[40-k] = \begin{cases} 1 & k = 20, 21, \ldots 60 \\ 0 & \text{else} \end{cases} \qquad x[k] = \begin{cases} \cos\left(\frac{\pi k}{4}\right) & |k| \le 40 \\ 0 & \text{else} \end{cases}$$

$$\hookrightarrow \tilde{X}(\omega, 40) = \sum_{k=20}^{40} \cos\left(\frac{\pi k}{4}\right) e^{-j\omega k} = \text{DTFT}\left\{ \cos\left[\frac{\pi k}{4}\right] \cdot w_2[k] \right\}$$

where $w_2[k] = \begin{cases} 1, & k = 20, 21, \ldots 40 \\ 0 & \text{else} \end{cases}$

~~$w_2[k] = w'[n-20] \Longleftrightarrow$ psinc~~

$$w_2'[k] = \begin{cases} 1 & k = 0, 1, \ldots 20 \\ 0 & \text{else} \end{cases} \overset{\text{DTFT}}{\longleftrightarrow} \text{psinc}_{21}(\omega) e^{-j\omega 10}$$

$$w_2[k] = w_2'[k-20] \overset{\text{DTFT}}{\longleftrightarrow} \text{psinc}_{21}(\omega) e^{-j\omega 10} e^{-j\omega 20}$$

$$= \text{psinc}_{21}(\omega) e^{-j\omega 30}$$

$$\tilde{X}(\omega, 40) = \text{DTFT}\left\{ \cos\left[\frac{\pi k}{4}\right] \cdot w_2[k] \right\}$$

$$= \boxed{\frac{1}{2} \text{psinc}_{21}\left(\omega - \frac{\pi}{4}\right) e^{-j\omega 30} + \frac{1}{2} \text{psinc}_{21}\left(\omega + \frac{\pi}{4}\right) e^{-j\omega 30}}$$

6 pts

See the following page for a correction to this answer.

iii $\boxed{n = 61}$

$$\tilde{X}(\omega, 61) = \sum_{k} x[k]\, w[61-k]\, e^{-j\omega k}$$

$$w[61-k] = \begin{cases} 1 & k = 41, 42, \ldots 81 \\ 0 & \text{else} \end{cases} \qquad x[k] = \begin{cases} \cos\left(\frac{\pi k}{4}\right) & |k| \le 40 \\ 0 & \text{else} \end{cases}$$

$$\hookrightarrow x[k] \cdot w[61-k] = 0$$

$$\hookrightarrow \tilde{X}(\omega, 61) = 0$$

5 pts

Correction to Solution for Exam 3 Problem 3(a)(ii)

The solutions to Problem 3(a)(i) and Problem 3(a)(iii) are correct. Only the solution to Problem 3(a)(ii) needs to be revised. Here $n = 40$; so we have

$$\begin{aligned}\tilde{X}(\omega,40) &= \text{DTFT}\left\{\cos\left(\frac{\pi k}{4}\right)\cdot w_2[k]\right\} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\pi\left(\delta(\mu-\frac{\pi}{4})+\delta(\mu-\frac{\pi}{4})\right)W_2(\omega-\mu)d\mu\,, \qquad (1) \\ &= \frac{1}{2}\left(W_2(\omega-\frac{\pi}{4})+W_2(\omega+\frac{\pi}{4})\right)\end{aligned}$$

where $W_2(\omega) = \text{DTFT}\left\{w_2[k]\right\}$ and $w_2[k] = \begin{cases} 1, & 20 \le k \le 40 \\ 0, & \text{else} \end{cases}$. We have that

$w_2[k] = w_2'[k-20]$, where $w_2'[k] = \begin{cases} 1, & 0 \le k \le 20 \\ 0, & \text{else} \end{cases}$. Thus, $W_2'(\omega) = \text{psinc}_{21}(\omega)e^{-j\omega 10}$.

So $W_2(\omega) = W_2'(\omega)e^{-j\omega 20}$, and $W_2(\omega) = \text{psinc}_{21}(\omega)e^{-j\omega 30}$. Substituting this last result into Eq. (1), we obtain

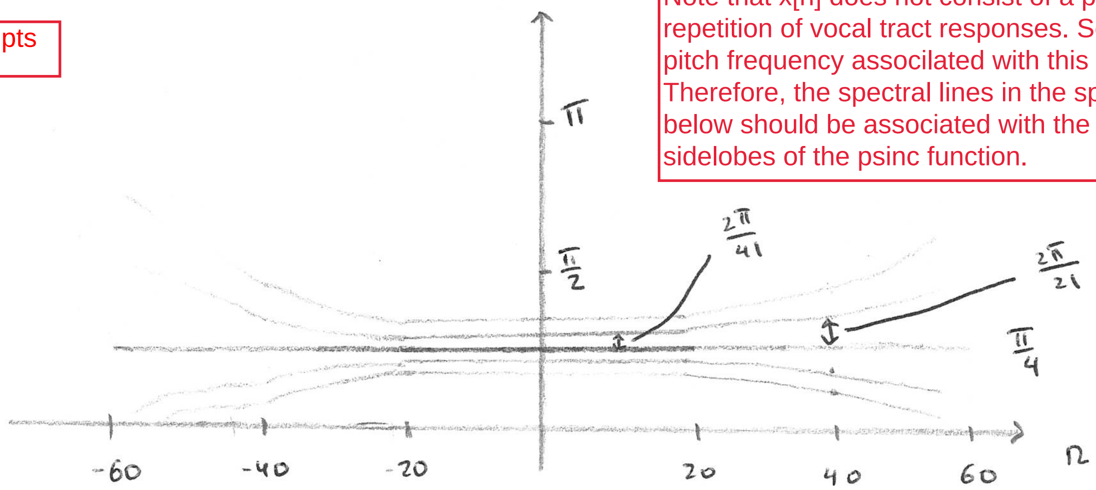
$$\tilde{X}(\omega,40) = \frac{1}{2}\left(\text{psinc}_{21}(\omega-\frac{\pi}{4})e^{-j(\omega-\frac{\pi}{4})30}+\text{psinc}_{21}(\omega+\frac{\pi}{4})e^{-j(\omega+\frac{\pi}{4})30}\right), \qquad (2)$$

which is a bit different from the originally posted solution shown on the previous page.

3. (continued - 2)

8 pts

Note that x[n] does not consist of a periodic repetition of vocal tract responses. So there is no pitch frequency associated with this waveform. Therefore, the spectral lines in the spectrogram below should be associated with the peaks in the sidelobes of the psinc function.



period = $\frac{2\pi}{\frac{\pi}{4}}$ = 8 samples

window length = 40 samples

} narrowband

4. (25)

Consider the signal $x(t)$s defined as follows

$$x(t) = t^2, 0 \le t \le 1$$

We wish to approximate $x(t)$ over the interval $0 \le t \le 1$ by the signal $\hat{x}(t)$ given by

$$\hat{x}(t) = a + bt,$$

where a and b are constants chosen to minimize

$$\varepsilon = \int_0^1 \left[\hat{x}(t) - x(t)\right]^2 dt.$$

a. (19) Determine the values for a and b that will minimize ε.

b. (6) Carefully sketch $x(t)$ and $\hat{x}(t)$ on the same axes for the optimal values for a and b that you determined in part (a) above.

a. $\varepsilon = \int_0^1 (a + bt - t^2)^2 \, dt$

$= \int_0^1 a^2 + b^2t^2 + t^4 + 2abt - 2at^2 - 2bt^3 \, dt$

$= \int_0^1 t^4 - 2bt^3 + (b^2 - 2a)t^2 + 2abt + a^2 \, dt$

$= \frac{1}{5}t^5 - \frac{1}{2}bt^4 + \frac{1}{3}(b^2 - 2a)t^3 + abt^2 + a^2t \Big|_0^1$

$= a^2 + (b - \frac{2}{3})a + \frac{1}{3}b^2 - \frac{1}{2}b + \frac{1}{5}$ +4

To find a and b that minimize ε, take derivatives. +3

$\frac{\partial \varepsilon}{\partial a} = 2a + (b - \frac{2}{3}) = 0$

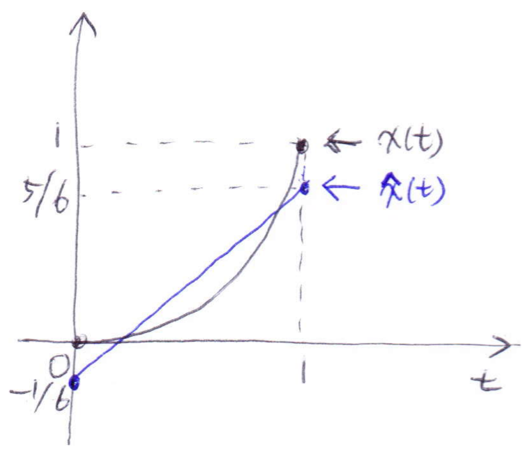
$\frac{\partial \varepsilon}{\partial b} = \frac{2}{3}b + (a - \frac{1}{2}) = 0$ +7

Solving the equations gives $a = -\frac{1}{6}$ and $b = 1$. +5

4. (continued - 1)

- Knows the strategy to minimize the error : take the derivative with regard to a and b. (+7)
- Correct procedure ~~to~~ or work to take the partial derivatives. (+7)
- Find correct a and b. (+5)

b.



- Shape of the two graphs. (+2)
- Dimension all important quantities. (+2)
- Geometrical relation between the two graphs. e.g., there are two crossings. (+2)

Partial credits

- Only the x(t) looks correct and has dimensions. (+1)
- Meets all three requirements described above but based on wrong answer in (a). (+3)