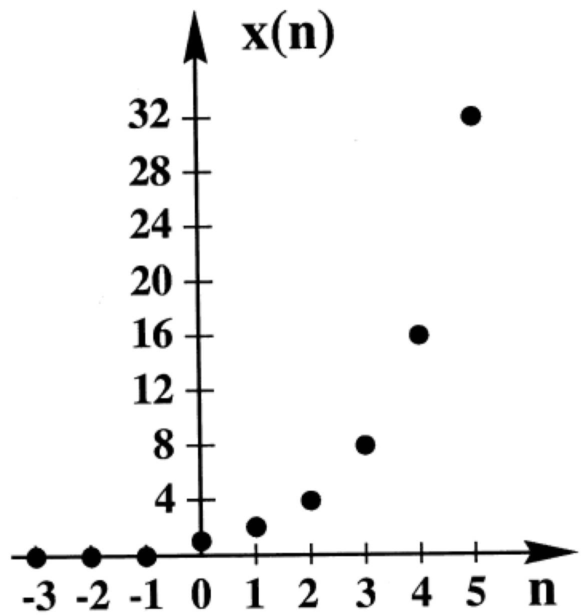
Z Transform (ZT)

Definition of Z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Example 1

Consider $x(n) = 2^n u(n)$



- It also satisfies neither condition for existence of the DTFT.

Recall formula for geometric series

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^{N-1}}{1-a}$$

for any complex number a and any finite integer N.

What happen as $N \to \infty$?

The sum remains finite provided that $|a| < 1$, since $a^{N-1} \to 0$, as $N \to \infty$

Thus, we have

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \text{, if } |a| < 1$$

For the example $x(n) = 2^n\ u(n)$,

we have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(2z^{-1}\right)^n$$

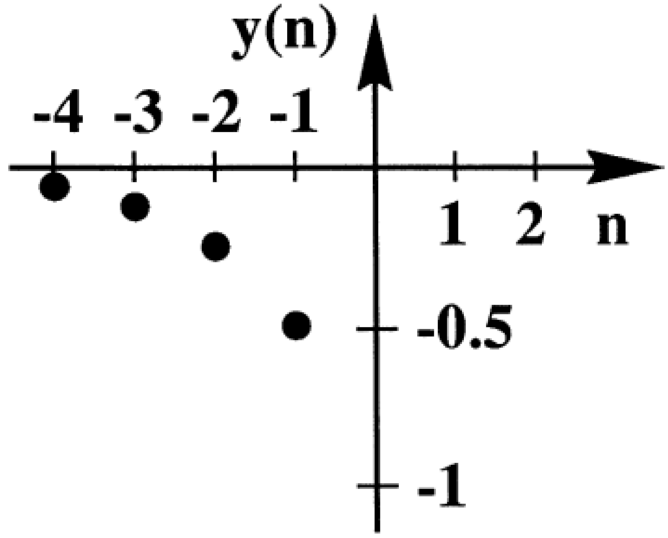
$$= \frac{1}{1-2z^{-1}}, \left|2z^{-1}\right| < 1$$

or $|z| > 2$.

- It is important to specify the region of convergence since the transform is not uniquely defined without it.

Example 2

Let $y(n) = -2^n\ u(-n-1)$



$$Y(z) = \sum_n y(n)\ z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} 2^n\ z^{-n}$$

$$Y(z) = -\sum_{n=-\infty}^{-1} (z/2)^{-n}$$

$$= -\sum_{n=1}^{\infty} (z/2)^{n}$$

$$= -\sum_{n=0}^{\infty} (z/2)^{n} + 1$$

$$Y(z) = 1 - \frac{1}{1 - z/2} \quad , \quad |\, z/2 \,| < 1$$

$$\text{or} \quad |\, z \,| < 2$$

$$= \frac{-z/2}{1 - z/2}$$

$$= \frac{1}{1 - 2z^{-1}} \quad , \quad |\, z \,| < 2$$

so we have

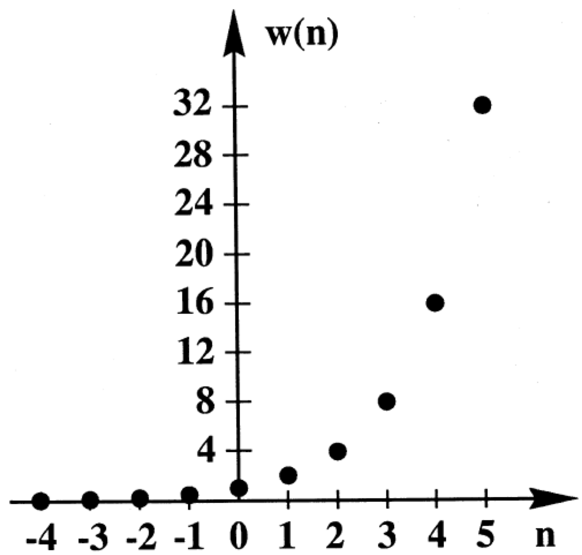
$$x(n) = 2^n u(n) \overset{ZT}{\longleftrightarrow} X(z) = \frac{1}{1 - 2z^{-1}} \quad , \quad |z| > 2$$

$$y(n) = -2^n \; u(-n-1) \overset{ZT}{\longleftrightarrow} Y(z) = \frac{1}{1 - 2z^{-1}} \; , \; |z| < 2$$

- The two transforms have the same functional form.
- They differ only in their regions of convergence.

Example 3

$w(n) = 2^n, \quad -\infty < n < \infty$



$$w(n) = x(n) - y(n)$$

By linearity of the ZT,

$$W(z) = X(z) - Y(z)$$

$$= \frac{1}{1 - 2z^{-1}} - \frac{1}{1 - 2z^{-1}} = 0\ !$$

- But note that X(z) and Y(z) have no common region of convergence.

  $\therefore$ There is no ZT for $w(n) = 2^n$, $-\infty < n < \infty$.

Z Transform Properties and Pairs

Transform Relations

1. linearity

$$a_1x_1(n) + a_2x_2(n) \overset{ZT}{\longleftrightarrow} a_1X_1(z) + a_2X_2(z)$$

2. shifting

$$x(n - n_0) \overset{ZT}{\longleftrightarrow} z^{-n_0}X(z)$$

3. modulation

$$z_0^n\, x(n) \overset{ZT}{\longleftrightarrow} X(z/z_0)$$

4. multiplication by time index

$$n\, x(n) \overset{ZT}{\longleftrightarrow} -z \frac{d}{dz} [X(z)]$$

5. convolution

$$x(n) * y(n) \overset{ZT}{\longleftrightarrow} X(z)\, Y(z)$$

6. relation to DTFT

If $X_{ZT}(z)$ converges for $|z| = 1$, then the DTFT of x(n) exists and

$$X_{DTFT}(e^{j\omega}) = X_{ZT}(z)\big|_{z=e^{j\omega}}$$

Important Transform Pairs

1. $\delta(n) \overset{ZT}{\longleftrightarrow} 1,$ all z

2. $a^n u(n) \overset{ZT}{\longleftrightarrow} \dfrac{1}{1 - az^{-1}},$ $|z| > a$

3. $-a^n u(-n-1) \overset{ZT}{\longleftrightarrow} \dfrac{1}{1 - az^{-1}},$ $|z| < a$

4. $na^n u(n) \overset{ZT}{\longleftrightarrow} \dfrac{az^{-1}}{(1 - az^{-1})^2},$ $|z| > a$

5. $-na^n u(-n-1) \overset{ZT}{\longleftrightarrow} \dfrac{az^{-1}}{(1 - az^{-1})^2},$ $|z| < a$

1.5.4 ZT AND LINEAR, CONSTANT COEFFICIENT DIFFERENCE EQUATIONS

- From the convolution property, we obtain a characterization for all LTI systems

  - impulse response: $y(n) = h(n) * x(n)$
  - transfer function: $Y(z) = H(z)\,X(z)$

- An important class of LTI systems are those characterized by linear, constant coefficient difference equations

$$y(n) = \sum_{k=0}^{M} a_k\, x(n-k) - \sum_{\ell=1}^{N} b_\ell\, y(n-\ell)$$

  - nonrecursive
    $N = 0$
    always finite impulse response (FIR)
  - recursive
    $N > 0$
    usually infinite impulse response (IIR)

- Take ZT of both sides of equation

$$y(n) = \sum_{k=0}^{M} a_k \, x(n-k) - \sum_{\ell=1}^{N} b_\ell \, y(n-\ell)$$

$$Y(z) = \sum_{k=0}^{M} a_k \, z^{-k} \, X(z) - \sum_{\ell=1}^{N} b_\ell \, z^{-\ell} \, Y(z)$$

Rearrange:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k \, z^{-k}}{1 + \sum_{\ell=1}^{N} b_\ell \, z^{-\ell}}$$

- By the fundamental theorem of algebra, the numerator and denominator polynomials may always be factored
  - $M \geq N$

$$H(z) = \frac{\prod_{k=1}^{M} (z - z_k)}{z^{M-N} \prod_{\ell=1}^{N} (z - p_\ell)}$$

- Roots of the numerator and denominator polynomials
  - zeros $z_1, \ldots, z_M$
  - poles $p_1, \ldots, p_N$

- The poles and zeros play an important role in determining system behavior.

Example

$$y(n) = x(n) + x(n-1) - \frac{1}{2}\, y(n-2)$$

$$Y(z) = X(z) + z^{-1}\, X(z) - \frac{1}{2}\, z^{-2}\, Y(z)$$

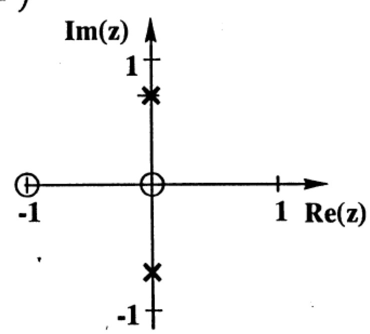
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 + \frac{1}{2}\, z^{-2}}$$

$$H(z) = \frac{z(z+1)}{z^2 + 1/2}$$

$$= \frac{z(z+1)}{(z - j/\sqrt{2})(z + j/\sqrt{2})}$$

zeros: $z_1 = 0$ $z_2 = -1$

poles: $p_1 = j/\sqrt{2}$ $p_2 = -j/\sqrt{2}$



Effect of Poles and Zeros on Frequency Response

Frequency response $H(e^{j\omega})$

$$h(n) \overset{\text{DTFT}}{\longleftrightarrow} H_{DTFT}(e^{j\omega}) = H(e^{j\omega})$$

$$h(n) \overset{\text{ZT}}{\longleftrightarrow} H_{ZT}(z)$$

$$H_{DTFT}(e^{j\omega}) = H_{ZT}(e^{j\omega})$$

Assume $M < N$

$$H(z) = \frac{z^{N-M} \prod_{k=1}^{M} (z - z_k)}{\prod_{\ell=1}^{N} (z - p_\ell)}$$

$$H(e^{j\omega}) = \frac{e^{j\omega(N-M)} \prod_{k=1}^{M} (e^{j\omega} - z_k)}{\prod_{\ell=1}^{N} (e^{j\omega} - p_\ell)}$$

$$|H(e^{j\omega})| = \frac{\prod_{k=1}^{M} |e^{j\omega} - z_k|}{\prod_{\ell=1}^{N} |e^{j\omega} - p_\ell|}$$

$$\angle H(e^{j\omega}) = \omega(N - M) + \sum_{k=1}^{M} \underline{/e^{j\omega} - z_k}$$

$$- \sum_{\ell=1}^{N} \underline{/e^{j\omega} - p_\ell}$$

Example

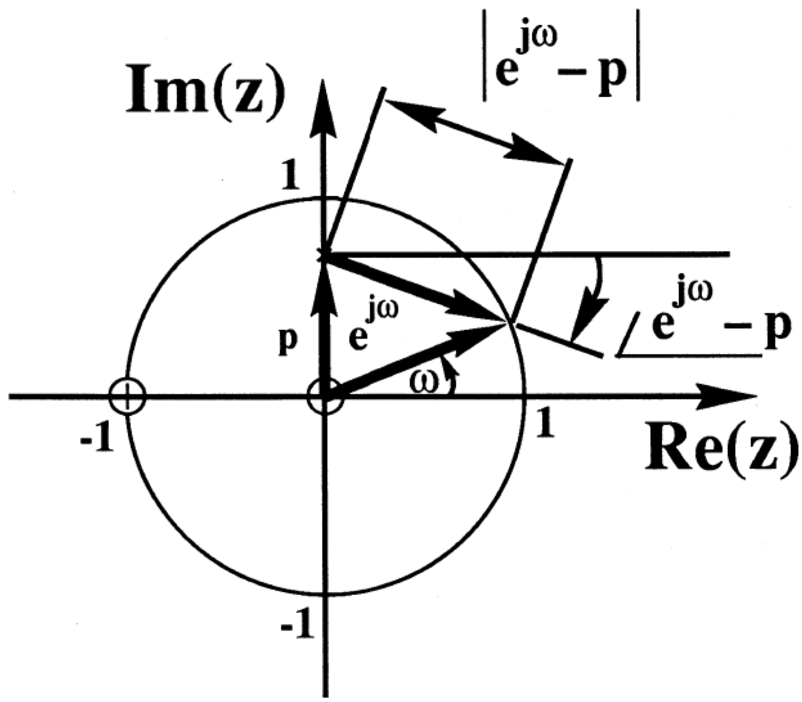
$$H(z) = \frac{z(z+1)}{(z - j/\sqrt{2})(z + j/\sqrt{2})}$$

$$|\,H(e^{j\omega})\,| = \frac{|\,e^{j\omega} + 1\,|}{|\,e^{j\omega} - j/\sqrt{2}\,|\ \ |\,e^{j\omega} + j/\sqrt{2}\,|}$$

$$\angle\ H(e^{j\omega}) = \omega + \underline{/e^{j\omega} + 1} - \underline{/e^{j\omega} - j/\sqrt{2}}$$

$$- \underline{/e^{j\omega} + j/\sqrt{2}}$$

Contribution from a single pole



ω = 0

$$\left|H(e^{j0})\right| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$$

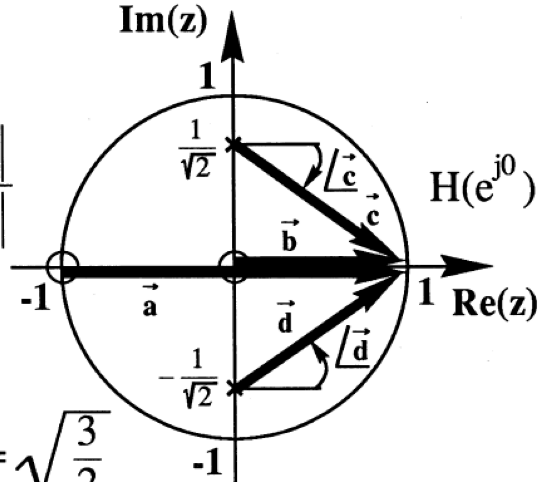
$$|\vec{a}| = 2$$

$$|\vec{b}| = 1$$

$$|\vec{c}| = \sqrt{1+\frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$|\vec{d}| = \sqrt{\frac{3}{2}}$$

$$\left|H(e^{j0})\right| = \frac{4}{3} = 1.33$$



$$\angle H(e^{j0}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$$

$$\angle\vec{a} = 0$$

$$\angle\vec{b} = 0$$

$$\angle\vec{c} = \arctan\left(\frac{1}{\sqrt{2}}\right)$$

$$\angle\vec{d} = -\angle\vec{c}$$

$$\angle H(e^{j0}) = 0$$

$\omega = \pi / 4$

$$\left|H(e^{j\pi/4})\right| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$$

$$|\vec{a}| = \sqrt{\left(1 + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$= 1.85$$

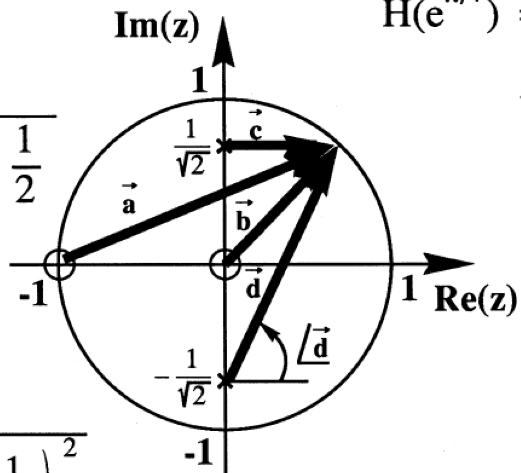
$$|\vec{b}| = 1$$

$$|\vec{c}| = \frac{1}{\sqrt{2}}$$

$$|\vec{d}| = \sqrt{\left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{5}{2}}$$

$$\left|H(e^{j\pi/4})\right| = 1.65$$



$$\angle H(e^{\pi/4}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$$

$$\angle\vec{a} = 22.5°$$

$$\angle\vec{b} = 45°$$

$$\angle\vec{c} = 0$$

$$\angle\vec{d} = 63.4°$$

$$\angle H(e^{j\pi/4}) = 4.1°$$

$\omega = \pi/2$

$$\left|H(e^{j\pi/2})\right| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$$

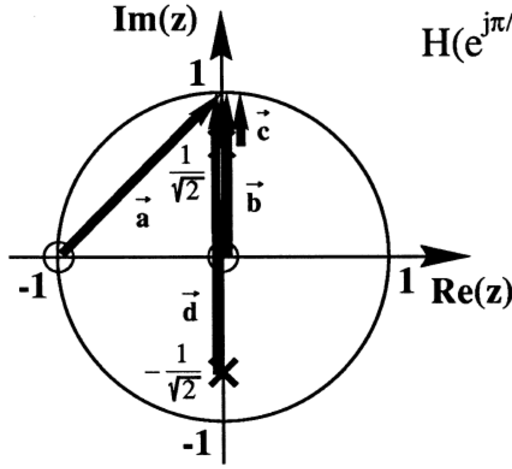
$$|\vec{a}| = \sqrt{2}$$

$$|\vec{b}| = 1$$

$$|\vec{c}| = 1 - \frac{1}{\sqrt{2}}$$

$$|\vec{d}| = 1 + \frac{1}{\sqrt{2}}$$

$$\left|H(e^{j\pi/2})\right| = 2.83$$



$$\angle H(e^{j\pi/2}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$$

$$\angle\vec{a} = 45°$$

$$\angle\vec{b} = 90°$$

$$\angle\vec{c} = 90°$$

$$\angle\vec{d} = 90°$$

$$\angle H(e^{j\pi/2}) = -45°$$

General Rules

- A pole near the unit circle will cause the frequency response to increase in the neighborhood of that pole.
- A zero near the unit circle will cause the frequency response to decrease in the neighborhood of that zero.