ECE 438 Lecture Notes 2/12/2016 (1)

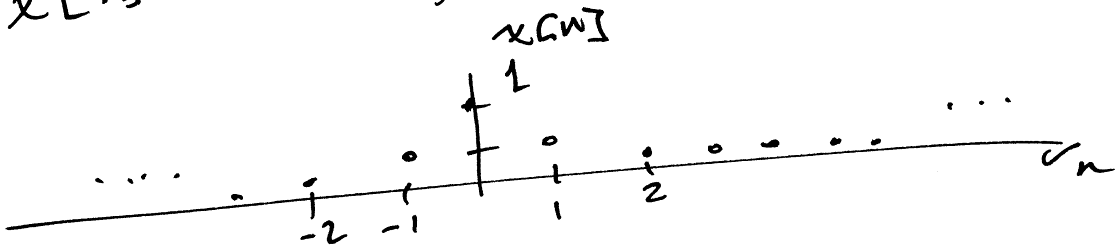
Signal Types

I. Infinite duration vs finite duration

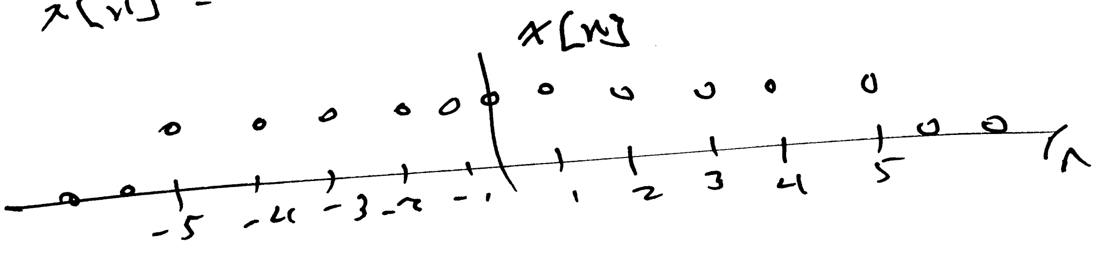
If $\exists$ ~~##~~ integers n_l & n_u $\ni$
(there exist) (such that)
$x[n] \neq 0$ only for $n_l \leq n \leq n_u$, then
$x[n]$ has finite-duration. Otherwise
$x[n]$ has infinite duration.

Examples

(1) $x[n] = e^{-|n|}, \quad -\infty < n < \infty$



(2) $x[n] = u[n+5] - u[n-6]$



II. Causal vs. ~~Non~~ anti causal vs. mixed causal (signals not systems)

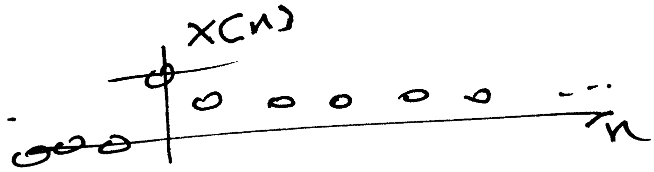
A. causal: $x[n] = 0, \quad n < 0$

B. anticausal: $x[n] = 0, \quad n > 0$

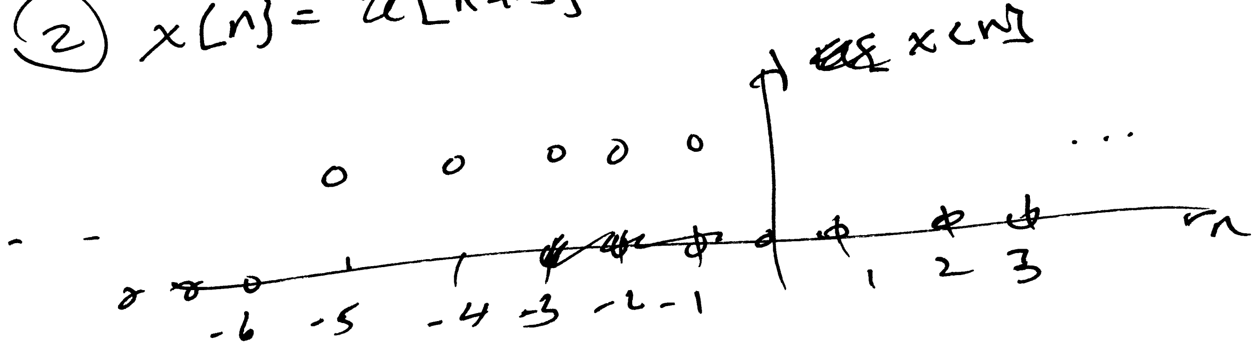
C. mixed causal: $\exists \; n_1 < 0$ & $n_2 > 0$ ∴
$x[n_1] \neq 0$ & $x[n_2] \neq 0$

Examples

① $x[n] = e^{-n} u[n]$



② $x[n] = u[n+5] - u[n]$



③ $x[n] = e^{-|n|} \quad -\infty < n < \infty$

I. Infinite duration vs. Finite duration

A. Finite duration

B. Infinite duration

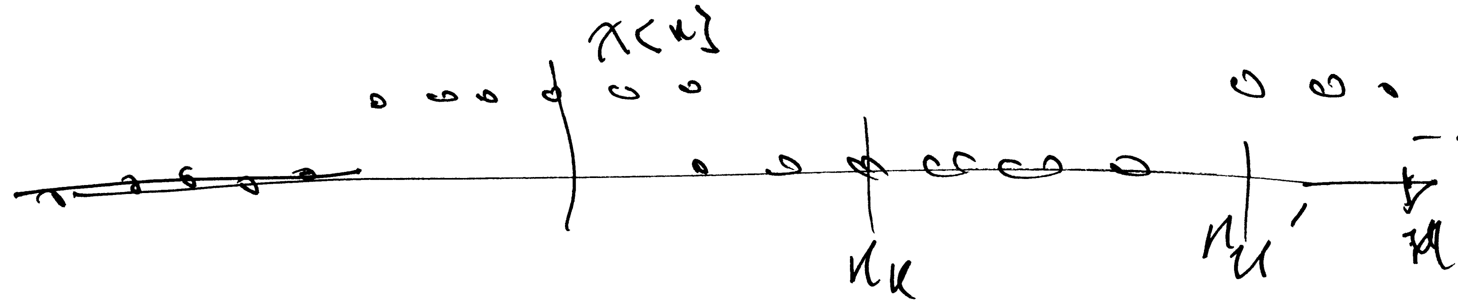
1. Right-sided (r.s.): $\exists\, n_\ell \ni x[n] = 0,$

$\forall\, n < n_\ell$

for all

Note also that by virtue of fact that $x[n]$ is infinite-duration, for <u>any</u> fixed n_u

$\exists\, n_u' > n_u \ni x[n_u'] \neq 0$



<u>Comparison:</u>

Energizer Bunny vs. a right-sided signal

<u>Energizer Bunny</u> – Once it starts, it never stops.

<u>Right-sided Signal</u> – It can stop as often as it wants to. <u>But</u>, It always starts again!

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Let $x[n]$ be a r.s. signal:
Let n_ℓ be <u>minimum</u> $n \ni x[n] \neq 0$

i.e. $x[n] = 0 \quad \forall\, n < n_\ell$

<u>then</u>
if $n_\ell \geq 0$ $x[n]$ is <u>causal</u>
if $n_\ell < 0$ $x[n]$ is mixed causal
$x[n]$ can <u>never</u> be anticausal.

I. B. Infinite duration
(2.) <u>Left-sided</u> (l.s.) $\equiv \exists\, n_u \ni x[n] = 0,$
$\forall\, n > n_u$
l.s. is same as r.s. but just <u>flipped</u>
i.e. if $x[n]$ is r.s., then $y[n] = x[-n]$
is l.s.

Let n_u be <u>maximum</u> $n \ni x[n] \neq 0$
i.e. $x[n] = 0, \quad \forall\, n > n_u$

<u>then</u>
if $n_u \leq 0$ $x[n]$ is anticausal
if $n_u > 0$, $x[n]$ is mixed causal
$x[n]$ can <u>never</u> be causal

I. B. (Infinite duration signals)

3. Two-sided 2.s

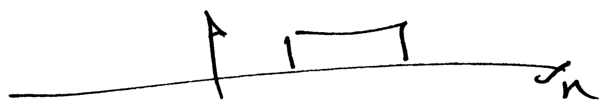
Given any integer n, $\exists\ n_1 < n < n_2$
$\ni\quad x[n_1] \neq 0$ & $x[n_2] \neq 0$

2.s. signals can only be mixed causal.
They cannot be ~~the~~ causal or anticausal.

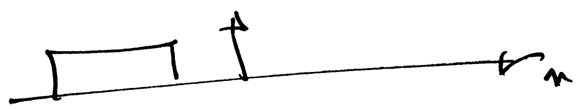
I. A. Finite-duration signals

Finite-duration ~~sig~~ signals can be:

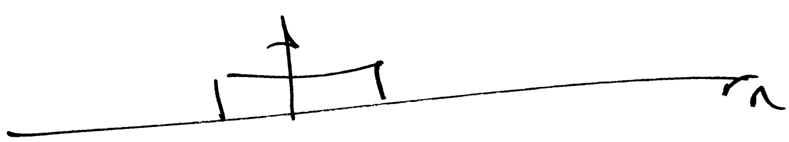
causal



anticausal



mixed causal



Summary

1. Every signal is either finite or infinite duration
2. Every infinite duration signal is either r.s., l.s., or 2.s.
3. Every signal is either causal, anticausal, or mixed causal

	Finite duration	r.s.	l.s.	2.s.
causal	✓	✓		
anticausal	✓		✓	
mixed causal	✓	✓	✓	✓

Infinite duration: r.s., l.s., 2.s.

Convergence of series

We have ~~definite~~ defined the DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

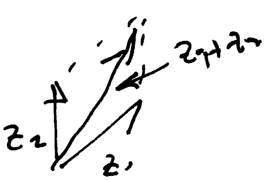
For each fixed $\underline{\omega}$ we want to know whether or not the series converges.
It is not very useful as a transform if it doesn't converge.

There are several ways to define convergence of a series:

$$S = \sum_{n=-\infty}^{\infty} a_n$$

~~We~~ Here we are interested in <u>absolute convergence</u>, i.e. we say that S converges iff ← if and only if $\equiv \Longleftrightarrow$

$$\hat{S} = \sum_{n=-\infty}^{\infty} |a_n| < \infty$$



Note that by triangle inequality $|z_1+z_2| \le |z_1|+|z_2|$ for any complex numbers z_1, z_2, z_3,

$$|S| \le \hat{S}$$

Z Transform (ZT)

We are finally ready to define the

ZT:

$$X_{ZT}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is any complex number

Compare to DTFT

$$X_{DTFT}(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

We see that ~~th~~

$$X_{ZT}(e^{j\omega}) = X_{DTFT}(\omega)$$

if $X_{ZT}(z)$ converges for $z = e^{j\omega}$.

We will drop subscript "ZT" in notation for ZT when there is no danger of confusion with the DTFT

i.e.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Convergence of ZT for finite duration signals:

$$X(z) = \sum_{n=n_l}^{n_u} x[n] z^{-n}$$

$$|X(z)| = \sum_{n=n_l}^{n_u} |x[n]| |z|^{-n}$$

Since there are only a finite number of terms, this has to converge unless

$|z| = 0$ (terms $n > 0$ blow up) or

$|z| = \infty$ (terms $n < 0$ blow up)

So, ZT of any finite duration signal ~~can be written~~ converges for

$$0 < |z| < \infty$$

Note: if x[n] is finite duration AND causal, then we have convergenced for 0 <= |z| < infinity.

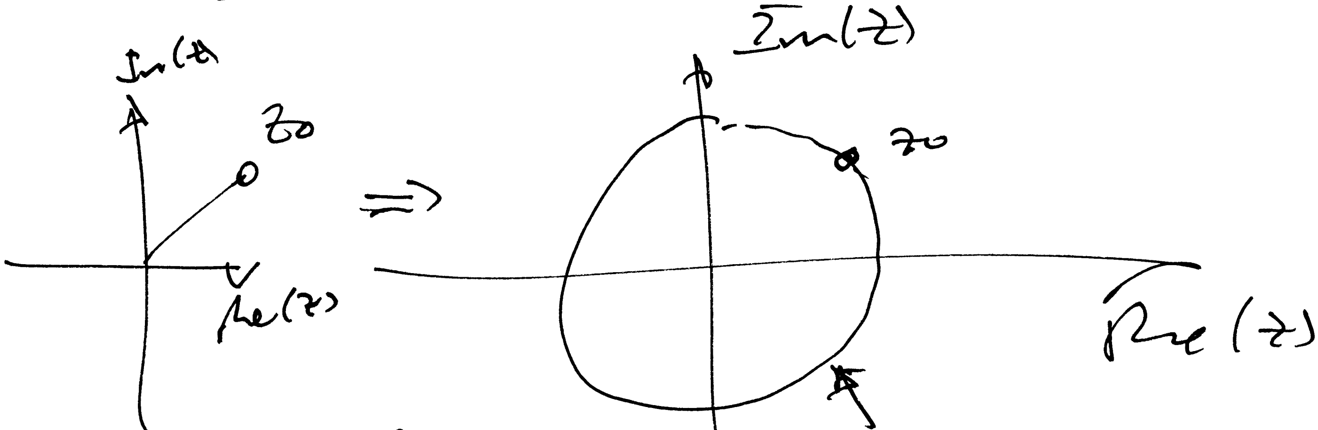
Convergence of ZT on a circle

Since we define convergence of ZT in terms of absolute convergence, we have the following:

Suppose ZT for $x[n]$ convergences at $z = z_0$. i.e.

$$\hat{X}(z_0) = \sum_n |x[n]| |z_0|^{-n} < \infty$$

Then $X(z)$ converges for all z s.t. $|z| = |z_0|$



convergence here $\Rightarrow$ convergence on circle $|z| = |z_0|$

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Convergence of ZT for r.s. signals

Note any r.s. signal $x[n]$ can be written as sum of ~~"acfd"~~ "acfd" anticausal finite duration r.s.s. signal and a causal r.s. signal "crs"

$$x_{rs}[n] = x_{acfd}[n] + x_{crs}[n]$$

where

$$x_{acfd}[n] = x_{rs}[n]\, u[-(n-1)]$$

$$x_{crs}[n] = x_{rs}[n]\, u[n]$$

Note: $u[-(n-1)] + u[n] \equiv 1$

Now ZT is linear

So $X_{rs}(z) = X_{acfd}(z) + X_{crs}(z)$

Also

$$ROC_{X_{rs}} = ROC_{X_{acfd}} \cap ROC_{X_{crs}}$$

here $ROC_X = \{z : X(z) \text{ converges}\}$

Convergence of ZT for c.r.s. signals

$$X_{crs}(z) = \sum_{n=n_\ell}^{\infty} x[n] z^{-n} \qquad n_\ell \geq 0$$

Since sum contains only negative powers of z and since

$$\cancel{|z_0^{-n}| < |z_1^{-n}|}$$

$$|z_0|^{-n} < |z_1|^{-n} \quad \forall n \geq 0$$

whenever $|z_1| > |z_0|$

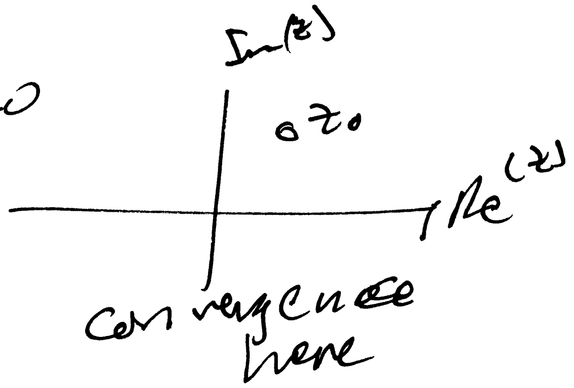
we have the following for a c.r.s.s.:

If $X_{crs}(z)$ converges for $z = z_0$

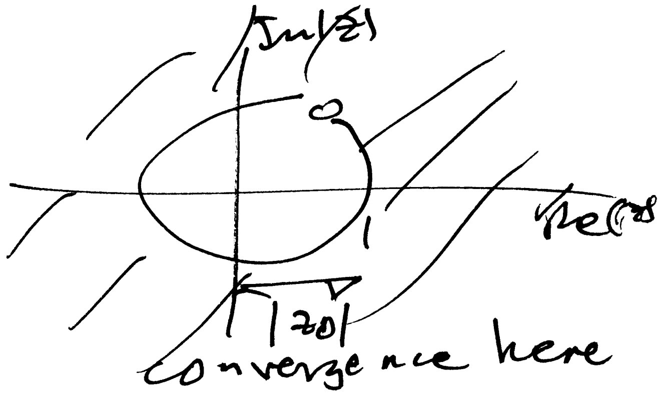
then $X_{crs}(z)$ converges for $\forall |z| \geq z_0$

i.e. $\hat{X}_{crs}(z_1) = \sum_n |x[n]| |z_1|^{-n} < \sum_n |x[n]| z_0^{-n} = \hat{X}_{crs}(z_0) < \infty$

so



$\Longrightarrow$



Convergence of ZT for r.s. signals (Cont.)

recall

$x_{rs}[n] = x_{acfd}[n] + x_{crs}[n]$

and $ROC_{X_{rs}} = ROC_{X_{acfd}} \cap ROC_{X_{crs}}$

$ROC_{X_{acfd}} = \{z : 0 < |z| < \infty\}$

$ROC_{X_{crs}} = \{z : |z| \geq r_0\}$

for some real $r_0 > 0$

$\therefore ROC_{X_{rs}} = \{z ; |z| \geq r_0\}$

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Convergence of ZT for l.s. signals

Recall $x_{ls}[n] = x_{rs}[-n]$

$\therefore \ ROC_{X_{ls}} = \{z : |z| \le r_0\}$

for some real $r_0 > 0$

Convergence of ZT for 2.s. signals

We can write

$$x_{2.s.}[n] = x_{l.s.}[n] + x_{r.s.}[n]$$

$$\therefore \ ROC_{X_{2s}} = ROC_{X_{ls}} \cap ROC_{X_{rs}}$$

$$= \{z : |z| \le r_{ls}\} \cap \{z : |z| \ge r_{rs}\}$$

$$= \begin{cases} \{z : r_{rs} \le |z| \le r_{ls}\}, & r_{rs} \le r_{ls} \\ \emptyset, & \text{else} \end{cases}$$

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Summary of ROCs for ZT

Boxes marked with "X"'s should be empty because these signals cannot exist.

	finite	r.s	l.s.	2.s
causal	$0<\|z\|<\infty$	$\|z\|>r_0$ or ϕ	$\|z\|<r_1$ or ϕ	$r_0<\|z\|<r_1$ or ϕ
anticausal	$\|z\|<\infty$ $0<$	~~$\|z\|>r_0$~~ $\|z\|>r_0$ or ϕ	$\|z\|<r_1$ or ϕ	$r_0<\|z\|<r_1$ or ϕ
mixed causal	$0<\|z\|<\infty$	~~$\|z\|>r_0$~~ $\|z\|>r_0$ or ϕ	$\|z\|<r_1$ or ϕ	$r_0<\|z\|<r_1$ or ϕ

r.s, l.s., 2.s: infinite