Continuous-Time Fourer Transform (CTFT)

Fourier Transform Pair

Forward transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \tag{1}$$

Inverse transform

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df \tag{2}$$

Transform Relations

1. linearity

$$a_1 x_1(t) + a_2 x_2(t) \overset{\text{CTFT}}{\leftrightarrow} a_1 X_1(f) + a_2 X_2(f)$$

2. scaling and shifting

$$x\left(\frac{t - t_0}{a}\right) \overset{\text{CTFT}}{\leftrightarrow} |a|\ X(af)\ e^{-j2\pi f t_0}$$

3. modulation

$$x(t)\ e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} X(f - f_0)$$

4. reciprocity

$$X(t) \overset{\text{CTFT}}{\leftrightarrow} x(-f)$$

5. Parseval's relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

6. Initial value

$$\int_{-\infty}^{\infty} x(t)dt = X(\mathbf{0})$$

Comments

1. Reflection is a special case of scaling and shifting with $a = -1$ and $t_0 = 0$, *i.e.*

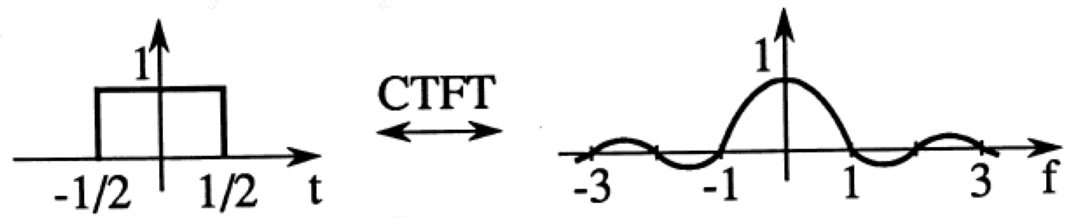
$$x(-t) \overset{\text{CTFT}}{\leftrightarrow} X(-f)$$

2. The scaling relation exhibits reciprocal spreading.

3. Uniqueness of the CTFT follows from Parseval's relation.

Important Transform Pairs

1. $\text{rect}(t) \overset{\text{CTFT}}{\longleftrightarrow} \text{sinc}(f)$

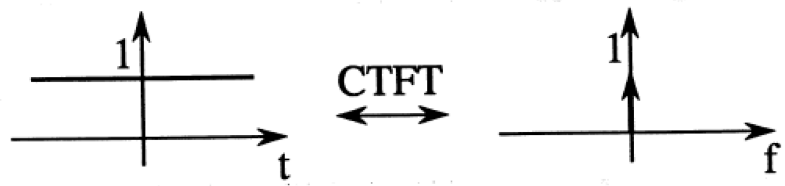


2. $\delta(t) \overset{\text{CTFT}}{\longleftrightarrow} 1$ (by sifting property)

Proof:

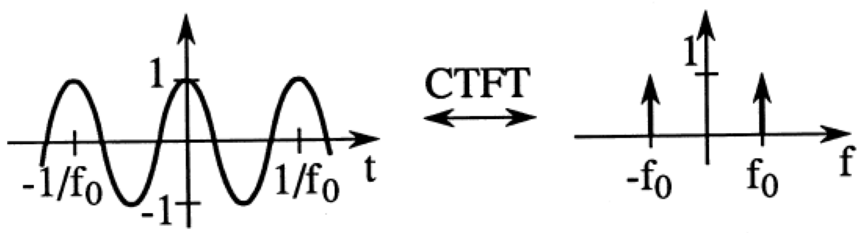
$$\mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft} dt = 1$$

3. $1 \overset{\text{CTFT}}{\leftrightarrow} \delta(f)$ (by reciprocity)



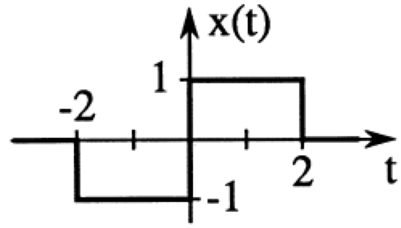
4. $e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} \delta(f - f_0)$ (by modulation property)

5. $\cos(2\pi f_0 t) \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$



Efficient Calculation of Fourier Transforms

Suppose we wish to determine the CTFT of the following signal



Brute force approach:

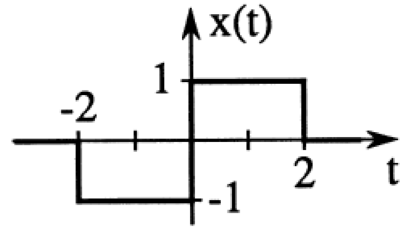
1. evaluate transform integral directly

$$X(f) = \int_{-2}^{0} (-1)\, e^{-j2\pi ft} dt + \int_{0}^{2} (1)\, e^{-j2\pi ft} dt$$

2. collect terms, simplify, etc...

Faster approach:

1. write x(t) in terms of functions whose transforms are known

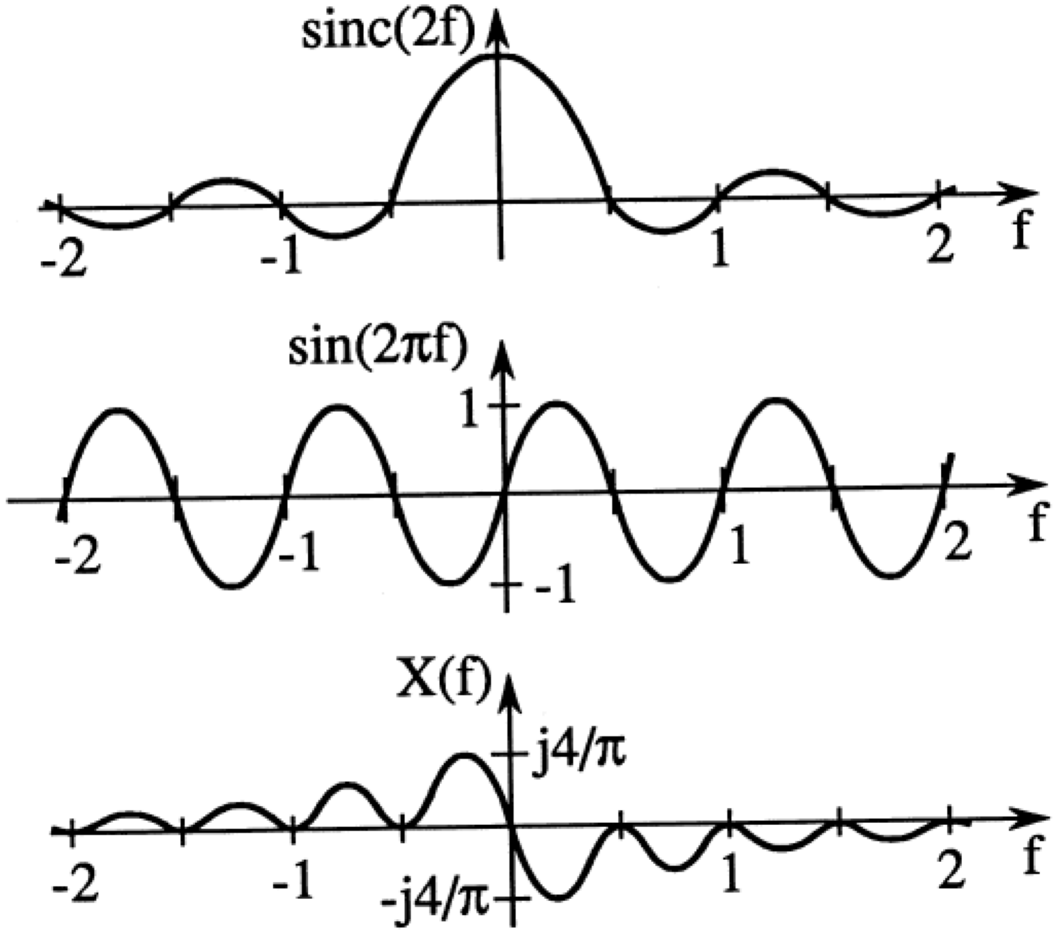


$$x(t) = -\,\mathrm{rect}\left(\frac{t+1}{2}\right) + \mathrm{rect}\left(\frac{t-1}{2}\right)$$

2. Use transform relations to determine X(f)

$$X(f) = 2\ \mathrm{sinc}(2f)\ [e^{-j2\pi f} - e^{j2\pi f}]$$

$$X(f) = -\, j4 \; \text{sinc}(2f) \; \sin(2\pi f)$$



Comments

1. $A_x = 0$ and $X(0) = 0$

2. x(t) is real and odd and X(f) is imaginary and odd

CTFT and LTI Systems – Two Equivalent Representations

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)\, x(\tau) d\tau$$

$$Y(f) = H(f)\, X(f)$$

where

$$x(t) \overset{\text{CTFT}}{\leftrightarrow} X(f)$$

$$h(t) \overset{\text{CTFT}}{\leftrightarrow} H(f)$$

$$y(t) \overset{\text{CTFT}}{\leftrightarrow} Y(f)$$

Convolution Theorem

Since x(t) and h(t) are arbitrary signals, we also have the following Fourier transform relation

$$\int x_1(\tau)\, x_2(t-\tau) d\tau \quad \overset{\text{CTFT}}{\leftrightarrow} \quad X_1(f)\, X_2(f)$$

or

$$x_1(t) * x_2(t) \quad \overset{\text{CTFT}}{\leftrightarrow} \quad X_1(f)\, X_2(f)$$

Product Theorem

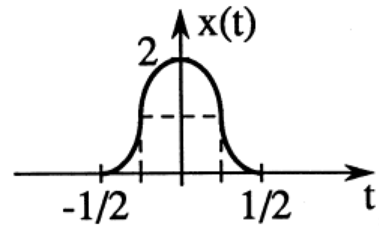
By reciprocity, we also have the following result

$$x_1(t)\, x_2(t) \quad \overset{\text{CTFT}}{\leftrightarrow} \quad X_1(f) * X_2(f)$$

This can be very useful for calculating transforms of certain functions.

Example

$$x(t) = \begin{cases} \frac{1}{2}\,[1 + \cos(2\pi t)] \ , & |\,t\,| \le 1/2 \\ 0 \ , & |\,t\,| > 1/2 \end{cases}$$



Find X(f)

$$x(t) = \frac{1}{2}\,[1 + \cos(2\pi t)]\ \text{rect}(t)$$

$$\therefore\ X(f) = \frac{1}{2}\left\{\delta(f) + \frac{1}{2}[\delta(f-1) + \delta(f+1)]\right\} * \text{sinc}(f)$$

Since convolution obeys linearity, we can write this as

$$X(f) = \frac{1}{2}\left\{\delta(f) * \text{sinc}(f) + \frac{1}{2}[\delta(f-1) * \text{sinc}(f) + \delta(f+1) * \text{sinc}(f)]\right\}$$

All three convolutions here are of the same general form.

Identity

For any signal w(t),

$$w(t) * \delta(t - t_0) = w(t - t_0)$$

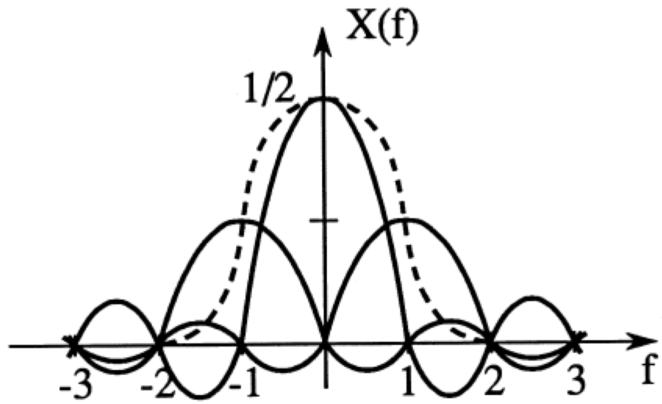
Proof:

$$\begin{aligned} w(t) * \delta(t - t_0) &= \int w(\tau)\, \delta(t - \tau - t_0) d\tau \\ &= w(t - t_0) \quad \text{(by sifting property)} \end{aligned}$$

Using the identity,

$$X(f) = \frac{1}{2}\left\{\delta(f) * \text{sinc}(f) + \frac{1}{2}[\delta(f-1) * \text{sinc}(f) + \delta(f+1) * \text{sinc}(f)]\right\}$$

$$= \frac{1}{2}\left\{\text{sinc}(f) + \frac{1}{2}\left[\text{sinc}(f-1) + \text{sinc}(f+1)\right]\right\}$$



CTFT of Periodic Signals and CTFT of Sampled Signals

Definitions:

$$\text{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\text{comb}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)$$

Transform Pairs:

$$\text{rep}_T\left[x(t)\right] \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{T}\text{comb}_{\frac{1}{T}}\left[X(f)\right]$$

$$\text{comb}_T\left[x(t)\right] \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{T}\text{rep}_{\frac{1}{T}}\left[X(f)\right]$$

The second relation follows from application of reciprocity to the first relation.

Example