Characterization of response of LTI systems

We now have:

1. Frequency Response

$$e^{j\omega_0 n} \overset{\text{SYS}}{\rightarrow} H(\omega_0) e^{j\omega_0 n}$$

2. DTFT: $$X(\omega) = \sum_n x[n] e^{-j\omega n}$$

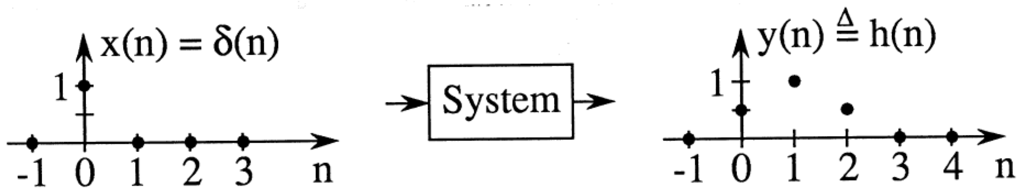
Inverse DTFT: $$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

3. Response of LTI System in Frequency Domain

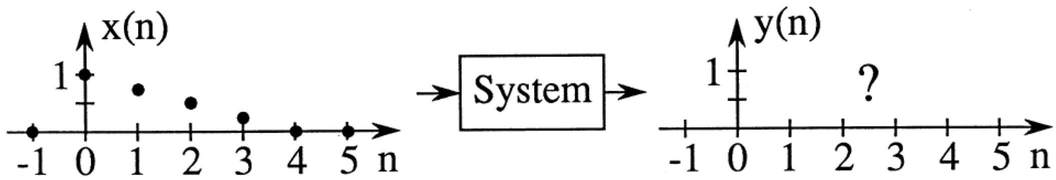
$$Y(\omega) = H(\omega) X(\omega)$$

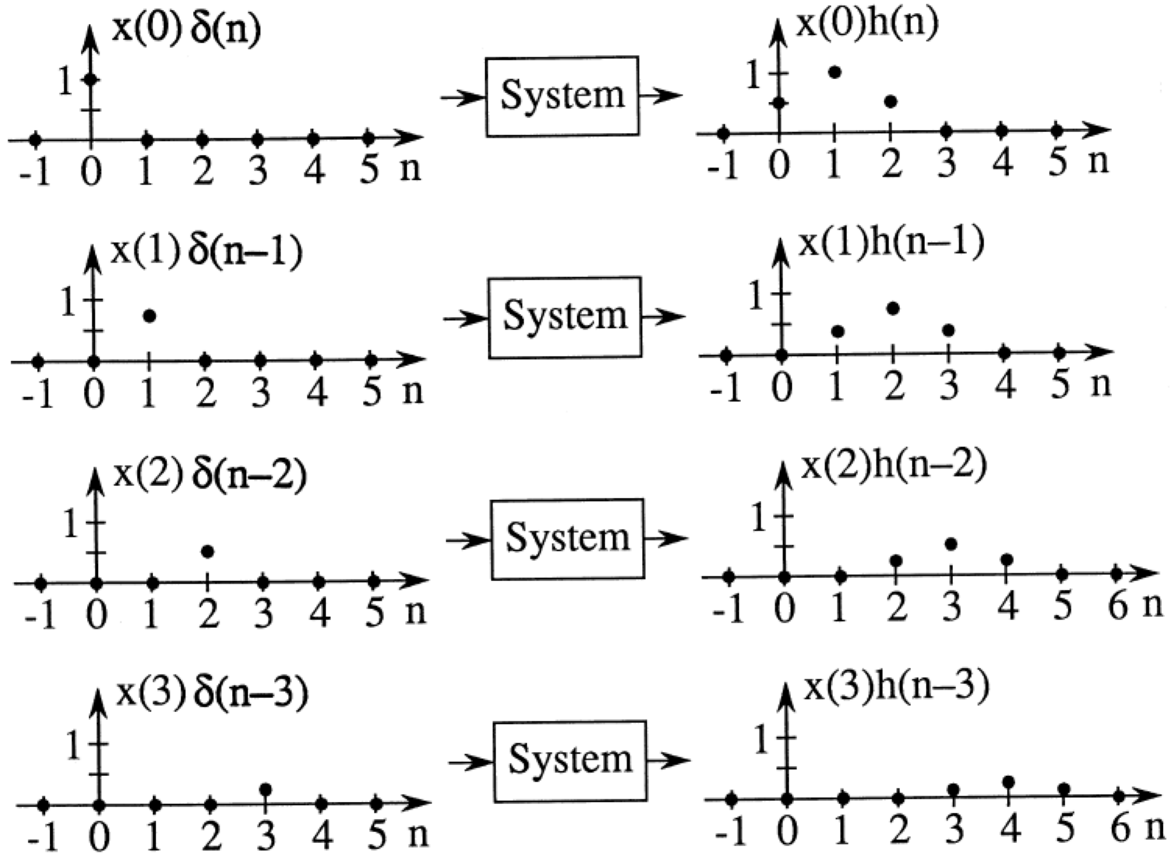
Next we consider an alternative **spatial domain** characterization:

Denote *impulse response* by h(n)

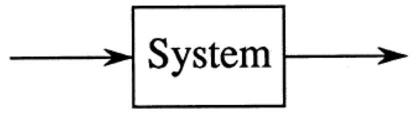


Now consider an arbitrary input x(n).



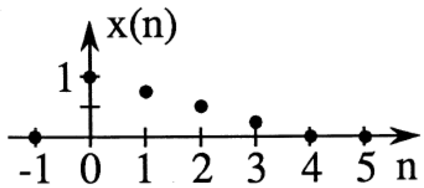

x(0) δ(n)
1
-1 0 1 2 3 4 5 n
System
x(0)h(n)
1
-1 0 1 2 3 4 5 n
x(1) δ(n–1)
1
-1 0 1 2 3 4 5 n
System
x(1)h(n–1)
1
-1 0 1 2 3 4 5 n
x(2) δ(n–2)
1
-1 0 1 2 3 4 5 n
System
x(2)h(n–2)
1
-1 0 1 2 3 4 5 6 n
x(3) δ(n–3)
1
-1 0 1 2 3 4 5 n
System
x(3)h(n–3)
1
-1 0 1 2 3 4 5 6 n

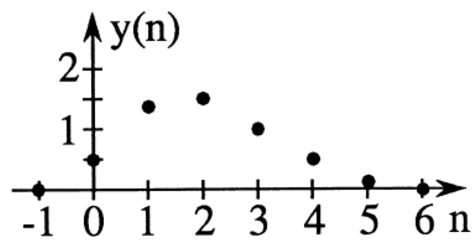
Sum over both the set of inputs and the set of outputs.



$$x(n) = x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + x(3)\,\delta(n-3)$$

$$y(n) = x(0)\,h(n) + x(1)\,h(n-1) + x(2)\,h(n-2) + x(3)\,h(n-3)$$





Convolution Sum

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

let $\ell = n - k \Rightarrow k = n - \ell$

$$y(n) = \sum_{\ell=\infty}^{-\infty} x(n-\ell)\,h(\ell) = \sum_{\ell=-\infty}^{\infty} x(n-\ell)\,h(\ell)$$

Notation and Identity

For any signals $x_1(n)$ and $x_2(n)$, we use an asterisk to denote their convolution; and we have the following identity

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(n-k)\, x_2(k)$$
$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\ .$$

Example:

DT System $\quad y(n) = \frac{1}{W} \sum_{k=0}^{W-1} x(n-k) \quad$ W - integer

Find response to $x(n) = e^{-n/D} u(n)$

W – width of averaging window

D – duration of input

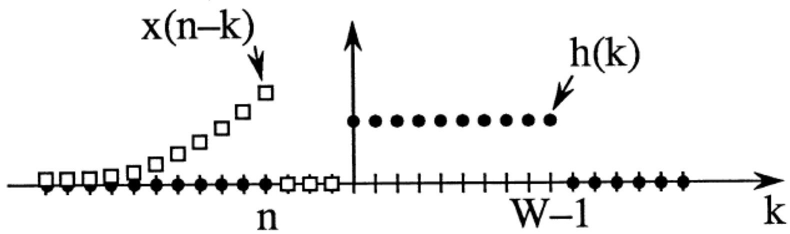
To find impulse response, let $x(n) = \delta(n) \Rightarrow h(n) = y(n)$

$$h(n) = \frac{1}{W} \sum_{k=0}^{W-1} \delta(n-k) = \begin{cases} 1/W, & 0 \leq n \leq W-1 \\ 0, & \text{else} \end{cases}$$

Now use convolution to find response to $x(n) = e^{-n/D} u(n)$.

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

Case 1: $n < 0$



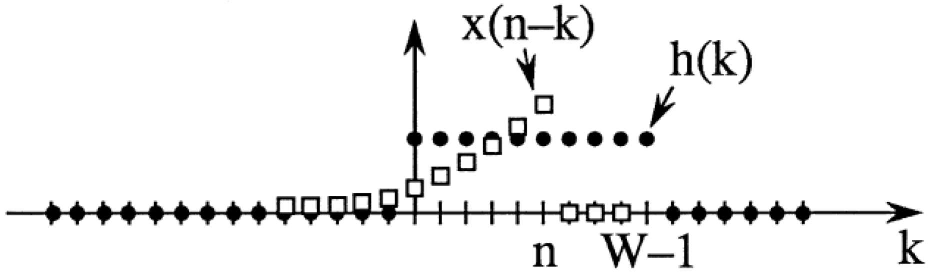
$y(n) = 0$

Note that:

$$x[n-k] = x[-(k-n)]$$

i.e., we **flip** the signal, **then shift** to center it at n.

Case 2: $0 \le n \le W-1$



$$y(n) = \sum_{k=0}^{n} x(n-k)\,h(k)$$

$$= \frac{1}{W} \sum_{k=0}^{n} e^{-(n-k)/D}$$

$$= \frac{1}{W} e^{-n/D} \sum_{k=0}^{n} e^{k/D}$$

Geometric Series

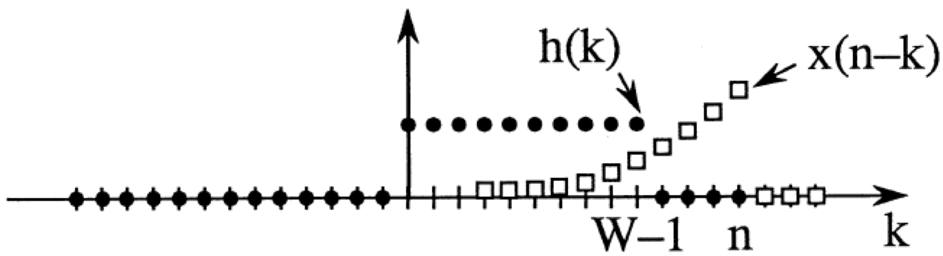
$$\sum_{k=0}^{N-1} z^k = \frac{1 - z^N}{1 - z}, \quad \text{for any complex number } z$$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}, \quad |z| < 1$$

$$y(n) = \frac{1}{W} e^{-n/D} \left[\frac{1 - e^{(n+1)/D}}{1 - e^{1/D}} \right]$$

$$= \frac{1}{W} \left[\frac{1 - e^{-(n+1)/D}}{1 - e^{-1/D}} \right]$$

Case 3: $W \le n$.



$$y(n) = \sum_{k=0}^{W-1} x(n-k)\,h(k)$$

$$= \frac{1}{W} \sum_{k=0}^{W-1} e^{-(n-k)/D}$$

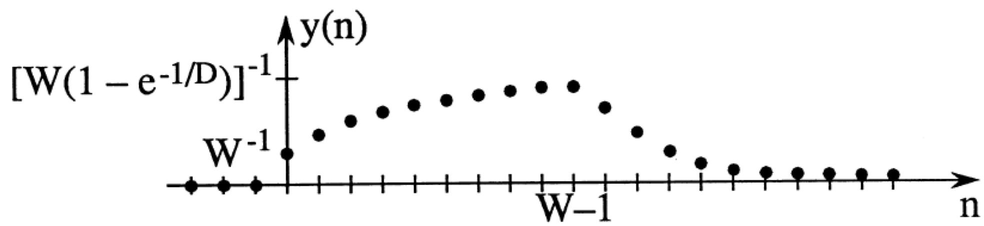
$$= \frac{1}{W}\, e^{-n/D} \sum_{k=0}^{W-1} e^{k/D}$$

$$y(n) = \frac{1}{W}\, e^{-n/D} \left[\frac{1 - e^{W/D}}{1 - e^{1/D}} \right]$$

$$= \frac{1}{W} \left[\frac{1 - e^{-W/D}}{1 - e^{-1/D}} \right] e^{-[n-(W-1)]/D}$$

Putting everything together

$$y(n) = \begin{cases} 0, & n < 0 \\ \dfrac{1}{W}\left[\dfrac{1 - e^{-(n+1)/D}}{1 - e^{-1/D}}\right], & 0 \le n \le W - 1 \\ \dfrac{1}{W}\left[\dfrac{1 - e^{-W/D}}{1 - e^{-1/D}}\right] e^{-[n-(W-1)]/D}, & W \le n \end{cases}$$



Causality for LTI Systems

$$y(n) = \sum_{k=-\infty}^{n} x(k)\,h(n-k) + \sum_{k=n+1}^{\infty} x(k)\,h(n-k)$$

contribution from past and present inputs	contribution from future inputs

System will be causal $\Leftrightarrow$ second sum is zero for any input $x(k)$.

This will be true $\Leftrightarrow h(n-k) = 0, \;\; k = n+1, \ldots, \infty$

$\Leftrightarrow h(k) = 0, \;\; k < 0$

$\therefore$ A LTI system is *causal* $\Leftrightarrow h(k) = 0,\ k < 0$,
i.e. the impulse response is a *causal signal*

Stability for LTI Systems

Suppose the input is bounded, *i.e.* $M_x < \infty$.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$|\, y(n) \,| = |\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\,|$$

$$\leq \sum_{k=-\infty} |\, x(k) \,| \; |\, h(n-k) \,|$$

$$= M_x \sum_{k=-\infty}^{\infty} |\, h(k) \,|$$

∴ It is sufficient for BIBO stability that the impulse response be absolutely summable.

Is it necessary?

Suppose $\sum_k |h(k)| \nless \infty$

Consider $y(0) = \sum_k x(k)\,h(-k)$

Assuming h(k) is real-valued, let

$$x(k) = \begin{cases} 1, & h(-k) > 0 \\ -1, & h(-k) < 0 \end{cases}$$

then $y(0) = \sum_k |h(k)| \nless \infty$

∴ A LTI system is *BIBO stable* ⇔ the impulse response is absolutely summable, *i.e.* $\sum_k |h(k)| < \infty$

Example

$$y(n) = x(n) + y(n-1)$$

Find the impulse response.

Let $x(n) = \delta(n)$, then $h(n) = y(n)$

Need to find solution to

$$y(n) = \delta(n) + 2\,y(n-1)$$

This example differs from earlier ones because the system is recursive, *i.e.* the current output depends on previous output values as well as the current and previous inputs.

1. must specify initial conditions for the system (assume $y(-1) = 0$).

2. cannot directly write a closed form expression for $y(n)$.

Find output sequence term by term

$$y(0) = \delta(0) + 2y(-1) = 1 + 2(0) = 1$$
$$y(1) = \delta(1) + 2y(0) = 0 + 2(1) = 2$$
$$y(2) = \delta(2) + 2y(1) = 0 + 2(2) = 4$$

Recognize general form

$$h(n) = y(n) = 2^n u(n)$$

1. Assuming system is initially at rest, it is causal.

2. $\sum_n |h(n)| \not< \infty \Rightarrow$ system is not BIBO stable.

Relation between Convolution and Frequency Response

For any DT LTI system, we know that input and output are related by

$$y(n) = \sum_{k} h(n-k)\, x(k)$$

Thus

$$Y(e^{j\omega}) = \sum_{n} y(n)\, e^{-j\omega n}$$

$$= \sum_{n} \sum_{k} h(n-k)\, x(k)\, e^{-j\omega n}$$

$$= \sum_{k} \{\sum_{n} h(n-k)\, e^{-j\omega n}\}\, x(k)$$

$$= \sum_{k} H(e^{j\omega})\, e^{-j\omega k}\, x(k) \quad \text{(by shifting property)}$$

where $H(e^{j\omega})$ is the DTFT of the impulse response $h(n)$.

Rearranging,

$$Y(e^{j\omega}) = H(e^{j\omega}) \sum_{k} x(k)\, e^{-j\omega k}$$

$$= H(e^{j\omega})\, X(e^{j\omega})$$