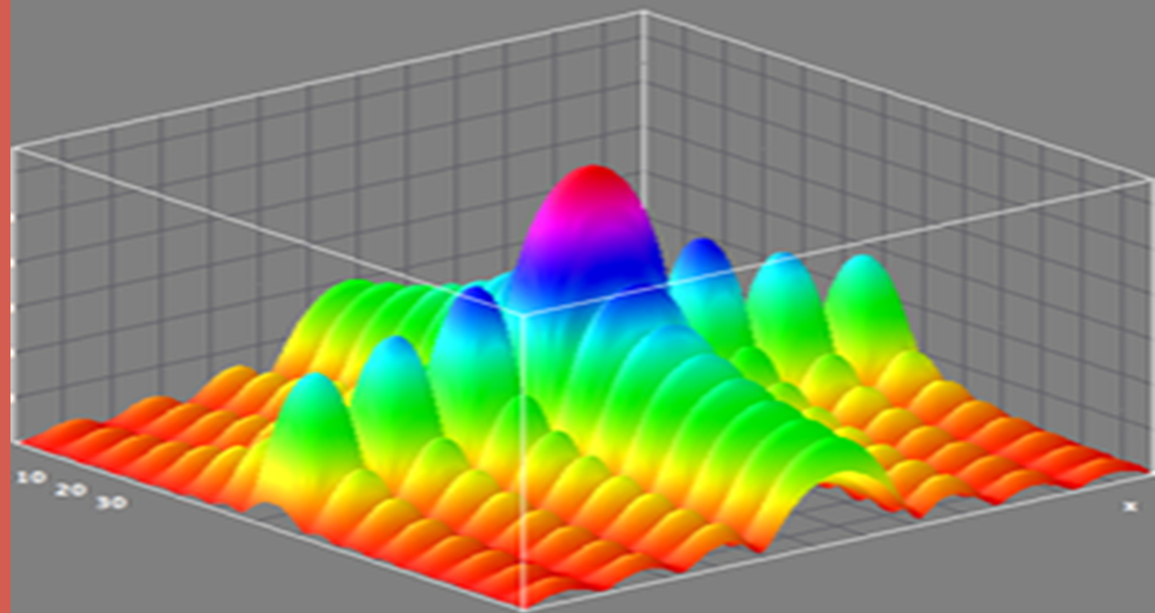
SECOND EDITION

DIGITAL SIGNAL PROCESSING IN A NUTSHELL VOLUME I

JAN ALLEBACH
KRITHIKA CHANDRASEKAR



Digital Signal Processing in a Nutshell (Volume I)

Jan Allebach, Krithika Chandrasekar

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Contents

Chapter 1

Signals

1.1 Chapter Outline

In this chapter, we will discuss:

1. Signal Types
2. Signal Characteristics
3. Signal Transformations
4. Special Signals
5. Complex Variables
6. Singularity Functions
7. Comb and Replication Operators

1.2 Signal Types

There are three different types of signals:

1. Continuous-time (CT)
2. Discrete-time (DT)
3. Digital (discrete-time and discrete-amplitude)

The different types of signals are shown in Figure 1.2(a) below.

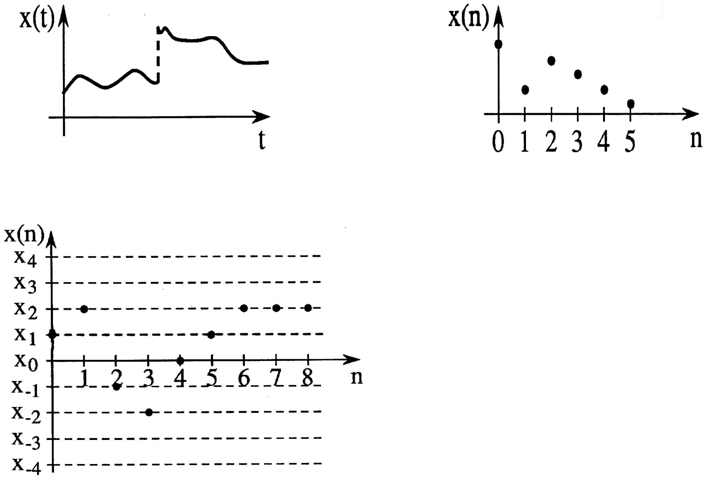


Figure 1.2(a) Different types of signals.

Comments

- The CT signal need not be continuous in amplitude
- The DT signal is undefined between sampling instances
- Levels need not be uniformly spaced in a digital signal
- We make no distinction between discrete-time and digital signals in theory
- The independent variable need not be time

- x(n) may or may not be sampled from an analog waveform, i.e.

  $x_d(n) = x_a(nT)$ T- sampling interval

  Throughout this book, we will use the subscripts d and a only when necessary for clarity.

Equivalent Representations for DT signals

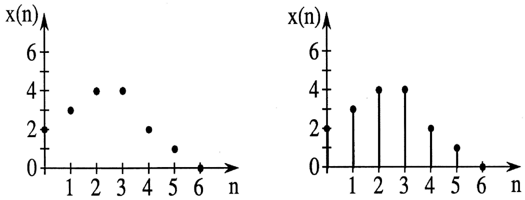


Figure 1.2(b) Equivalent representations for DT signals.

1.3 Signal Characteristics

Signals can be:

1. Deterministic or random
2. Periodic or aperiodic
   For periodic signals, $x(t) = x(t + nT)$, for some fixed T and for all integers n and times t.
3. Right-sided, left-sided, or two-sided
   For right-sided signals, $x(t) \neq 0$ only when $t \geq t_0$.

   If $t_0 \geq 0$, the signal is causal.
   For left-sided signals, $x(t) \neq 0$ only when $t \leq t_0$.

   If $t_0 \leq 0$, the signal is anticausal.
   For 2-sided or mixed causal signals, $x(t) \neq 0$ for some $t < 0$, and $x(t) \neq 0$ for some $t > 0$.
4. Finite or infinite duration
   For finite duration signals, $x(t) \neq 0$ only for $-\infty < t_1 \leq t \leq t_2 < \infty$.

Metrics

Metrics	CT signal	DT signal
Energy	$E_x = \int \lvert x(t)\rvert^2 dt$	$E_x = \sum_n \lvert x(n)\rvert^2$
Power	$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \lvert x(t)\rvert^2 dt$	$P_x = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} \lvert x(n)\rvert^2$
Magnitude	$M_x = \max_t \lvert x(t)\rvert$	$M_x = \max_n \lvert x(n)\rvert$
Area	$A_x = \int \lvert x(t)\rvert dt$	$A_x = \sum_n \lvert x(n)\rvert$
Average value	$x_{avg} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$	$x_{avg} = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} x(n)$

Comments

- If signal is periodic, we need only average over one period, e.g. if x(n)=x(n+N) for all n

$$P_x = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} |x(n)|^2, \text{ for any } n_0$$

- root-mean-square (rms) value

$$x_{rms} = \sqrt{P_x}$$

Examples

Example 1

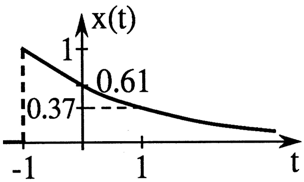


Figure 1.3(a) Signal for Example 1.

$$x(t) = \begin{cases} e^{-(t+1)/2}, & t > -1 \\ 0, & t < -1 \end{cases}$$

The signal is:

1. deterministic
2. aperiodic
3. right-sided
4. mixed causal

Metrics

1. $E_x = \int_{-1}^{\infty} |e^{-(t+1)/2}|^2 dt, \quad \text{let } s = t+1$

   $E_x = \int_0^{\infty} e^{-s} ds = -e^{-s}|_0^{\infty} = 1$
2. $P_x = 0$
3. $M_x = 1$
4. $A_x = 2$
5. $x_{\text{avg}} = 0$

Example 2

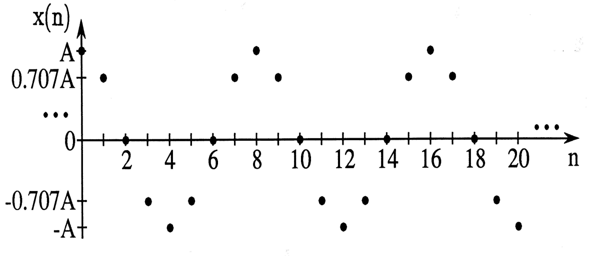


Figure 1.3(b) Signal for Example 2.

$x(n) = A\cos(\pi n/4)$

The signal is:

1. deterministic
2. periodic ($N = 8$)

3. 2-sided
4. mixed causal

Metrics

1. $E_x = \infty$
2. $P_x = \frac{1}{8}\sum_{n=0}^{7} |A\cos(\pi n/4)|^2$

$$P_x = \frac{A^2}{16}\sum_{n=0}^{7}[1+\cos(\pi n/2)]$$

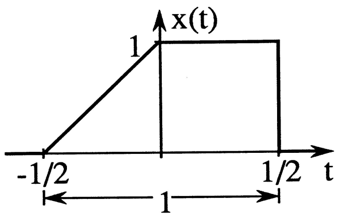
$$P_x = \frac{A^2}{2}$$

3. $x_{\text{rms}} = \frac{A}{\sqrt{2}}$
4. $M_x = A$
5. A_x is undefined
6. $x_{\text{avg}} = 0$

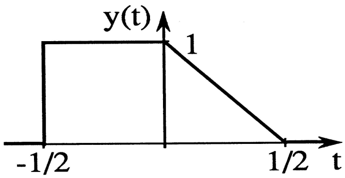
1.4 Signal Transformations

Continous-Time Case

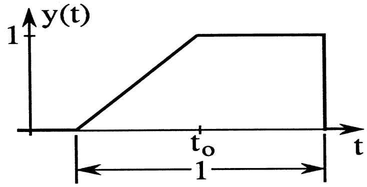
Consider



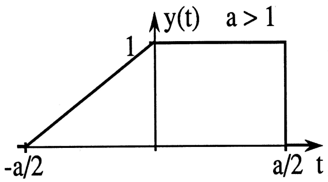
1. Reflection
   $y(t) = x(-t)$

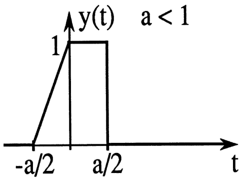


2. Shifting
   $y(t) = x(t - t_0)$



3. Scaling
   $y(t) = x(\frac{t}{a})$





4. Scaling and Shifting

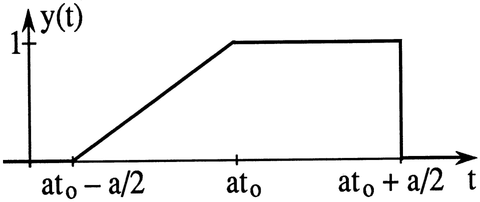
   <u>Example 1</u>

   $y(t) = x(\frac{t}{a} - t_0)$

   What does this signal look like?

   center of pulse: $\frac{t}{a} - t_0 = 0 \Rightarrow t = at_0$

left edge of pulse: $\frac{t}{a} - t_0 = \frac{-1}{2} \Rightarrow t = a(t_0 - \frac{1}{2})$
right edge of pulse: $\frac{t}{a} - t_0 = \frac{1}{2} \Rightarrow t = a(t_0 + \frac{1}{2})$



Example 2

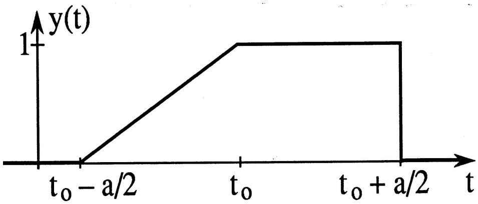
$y(t) = x\left(\frac{t-t_0}{a}\right)$

What does this signal look like?

center: $(t - t_0)/a = 0 \Rightarrow t = t_0$

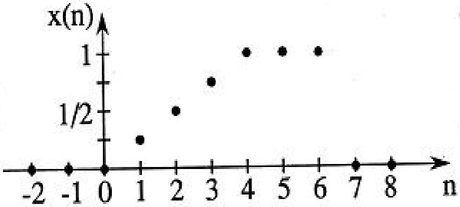
left edge of pulse: $(t - t_0)/a = -1/2 \Rightarrow t = t_0 - a/2$

right edge of pulse: $(t - t_0)/a = 1/2 \Rightarrow t = t_0 + a/2$

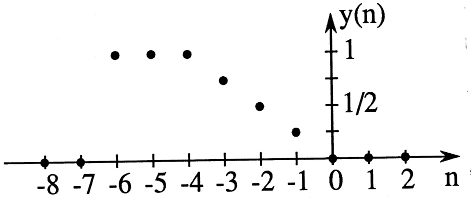


Discrete-Time Case

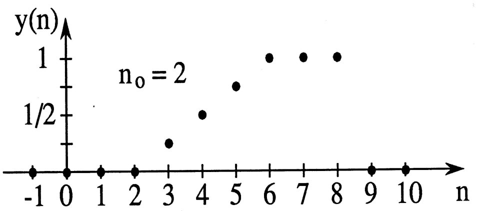
Consider



1. Reflection
   $y(n) = x(-n)$

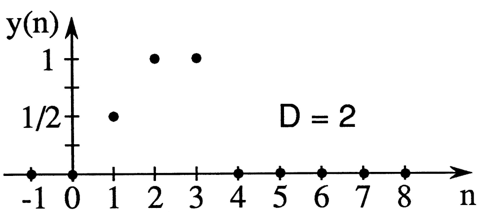


2. Shifting
   $y(n) = x(n - n_0)$

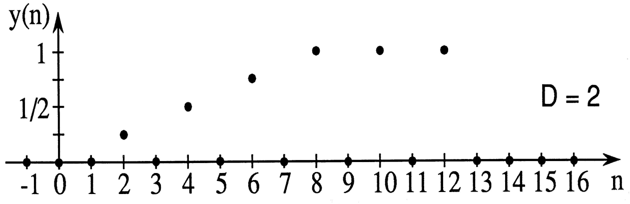


   Note that n_0 must be an integer. Non-integer delays can only be defined in the context of a CT signal, and are implemented by interpolation.

3. Scaling
   a. downsampler
   $y(n) = x(Dn), \; D \in \mathbb{Z}$



   b. upsampler
   $$y(n) = \begin{cases} x(n/D), & \text{n/D} \in \mathbb{Z},\ \text{D} \in \mathbb{Z} \\ 0, & \text{else} \end{cases}$$

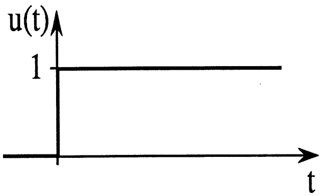


1.5 Special Signals

Unit step

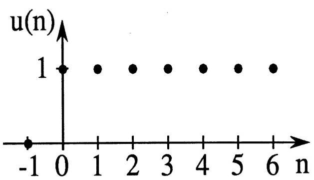
a. CT case

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



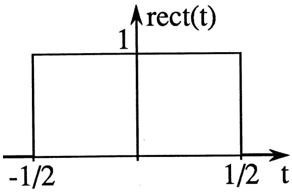
b. DT case

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & \text{else} \end{cases}$$



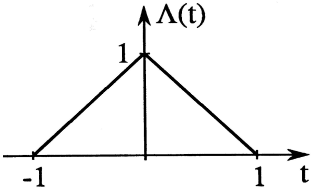
Rectangle

$$rect(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \end{cases}$$



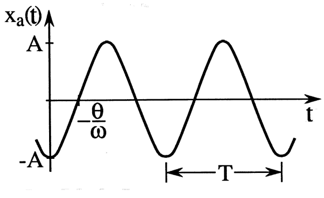
Triangle

$$\Lambda(t) = \begin{cases} 1 - |t|, & |t| \leq 1 \\ 0, & \text{else} \end{cases}$$



Sinusoids

a. CT



$x_a(t) = A\sin(\omega_a t + \theta)$
where A is the amplitude
ω_a is the frequency
θ is the phase

Analog frequency

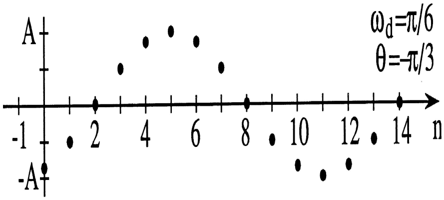
$$\omega_a \left(\frac{\text{radians}}{\text{sec}}\right) = 2\pi f_a \left(\frac{\text{cycles}}{\text{sec}}\right)$$

Period

$$T\left(\frac{\text{sec}}{\text{cycle}}\right) = \frac{1}{f_a}$$

b. DT

$x_d(n) = A\sin(\omega_d n + \theta)$

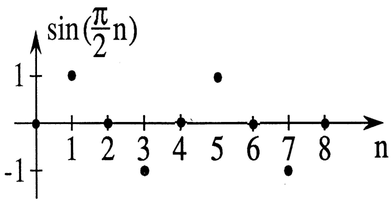


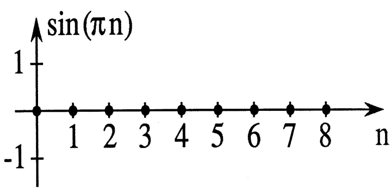
digital frequency

$$\omega_d\left(\frac{\text{radians}}{\text{sample}}\right) = 2\pi\mu\left(\frac{\text{cycles}}{\text{sample}}\right)$$

Comments

1. Depending on ω, $x_d(n)$ may not look like a sinusoid





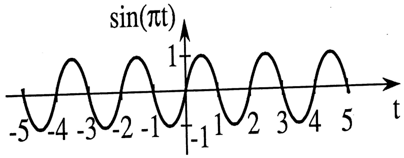
2. Digital frequencies $\omega_1 = \omega_0$ and $\omega_2 = \omega_0 + 2\pi k$ are equivalent

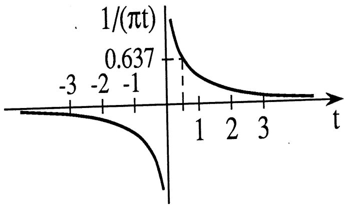
$$\begin{aligned} x_2(n) &= A\sin(\omega_2 n + \theta) \\ &= A\sin[(\omega_0 + 2\pi k)n + \theta] \\ &= x_1(n), \text{ for all n} \end{aligned}$$

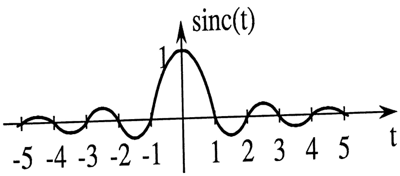
3. $x_d(n)$ will be periodic if and only if $\omega_d = 2\pi(p/q)$ where p and q are integers. In this case, the period is q.

Sinc

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{(\pi t)}$$







1.6 Complex Variables

Definition
Cartesian coordinates

$$z = (x, y)$$

Polar coordinates

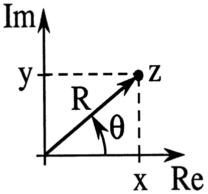
$$z = R\angle\theta$$

$$R = \sqrt{x^2 + y^2}$$

$$x = R\cos(\theta)$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$y = R\sin(\theta)$$



Alternate representation

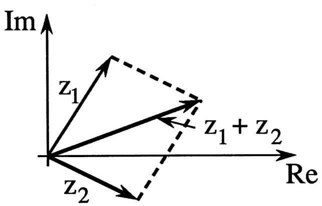
define $j = (0,1) = 1\angle(\pi/2)$
then $z = x + jy$

Additional notation

$Re\{z\} = x$
$Im\{z\} = y$
$|z| = R$

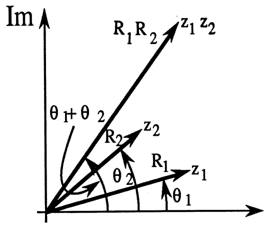
Algebraic Operations

Addition (Cartesian Coordinates)



$z_1 = x_1 + jy_1$
$z_2 = x_2 + jy_2$
$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

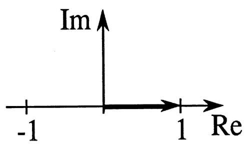
Multiplication (Polar Coordinates)



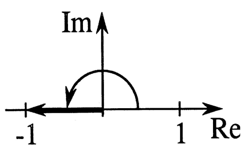
$z_1 = R_1\angle\theta_1$
$z_2 = R_2\angle\theta_2$
$z_1z_2 = R_1R_2\angle(\theta_1+\theta_2)$

Special Examples

1. $1 = (1, 0) = 1\angle 0$



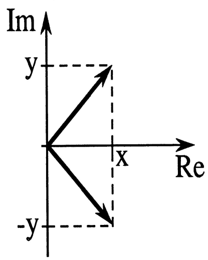
2. $-1 = (-1, 0) = 1\angle 180$



3. $j^2 = (1\angle 90)^2 = 1\angle 180 = -1$

Complex Conjugate

$z^* = x - jy = R\angle(-\theta)$



Some useful identities

$Re\{z\} = \frac{1}{2}[z + z^*]$

$Im\{z\} = \frac{1}{j2}[z - z^*]$

$|z| = \sqrt{zz^*}$

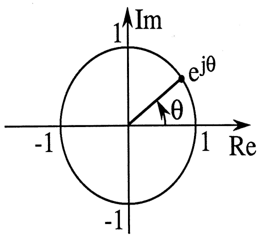
Complex Exponential

Taylor series for exponential function of real variable x

$$e^x = \sum_{\infty}^{k=0} \frac{x^k}{k!}$$

Replace x by jθ

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{j\theta^2}{2!} + \frac{j\theta^3}{3!} + \frac{j\theta^4}{4!} + \frac{j\theta^5}{5!} \\ &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...) + j(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...) \\ &= \cos\theta + j\sin\theta \end{aligned}$$



$|e^{j\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$
$\angle e^{j\theta} = \arctan\left(\frac{\sin\theta}{\cos\theta}\right) = \theta$

Alternate form for polar coordinate representation of complex number

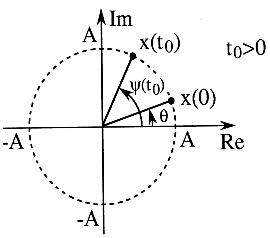
$$z = R\angle\theta = Re^{j\theta}$$

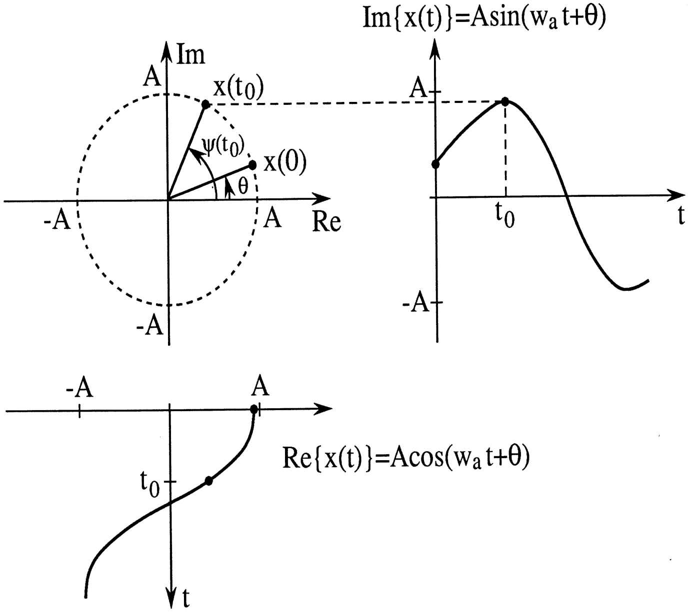
Multiplication of two complex numbers can be done using rules for multiplication of exponentials:

$$z_1 z_2 = (R_1 e^{j\theta_1})(R_2 e^{j\theta_2}) = R_1 R_2 e^{j(\theta_1+\theta_2)}$$

Complex exponential signal

CT $x(t) = Ae^{j\psi(t)}$
$\psi(t) = \omega_a t + \theta$ instantaneous phase
$\frac{d\psi(t)}{dt} = \omega_a \left(\frac{\text{rad}}{\text{sec}}\right)$ instantaneous phasor velocity



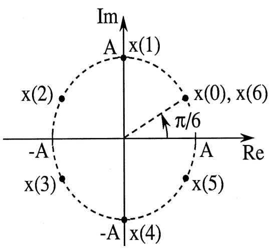


DT $x(n) = Ae^{j(\omega_d n+\theta)}$

ω_d is the phasor velocity in $\left(\frac{\text{radians}}{\text{sample}}\right)$.

Example

$\omega_d = \pi/3 \quad \theta = \pi/6$



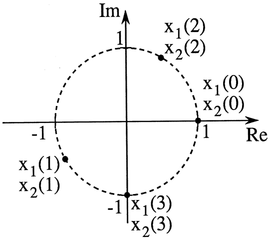
Aliasing

1. It refers to the ability of one frequency to mimic another
2. For CT case, $e^{j\omega_1 t} = e^{j\omega_2 t}$ for all $t \Leftrightarrow \omega_1 = \omega_2$
3. In DT, $e^{j\omega_1 n} = e^{j\omega_2 n}$ if $\omega_2 = \omega_1 + 2\pi k$

Consider

$\omega_1 = \frac{7\pi}{6} \quad \omega_2 = \frac{-5\pi}{6}$

$x_1(n) = e^{j\omega_1 n} \quad x_2(n) = e^{j\omega_2 n}$



In general, if $\omega_1 = \pi + \Delta$ and $\omega_2 = -(\pi - \Delta)$, then

$$\begin{aligned} x_1(n) &= e^{j\omega_1 n} \\ &= e^{j(\pi+\Delta)n} \\ &= e^{j[(\pi+\Delta)n - 2\pi n]} \\ &= e^{j[(-\pi+\Delta)n]} \\ &= e^{-j(\pi-\Delta)n} \\ &= x_2(n) \end{aligned}$$

1.7 Singularity functions

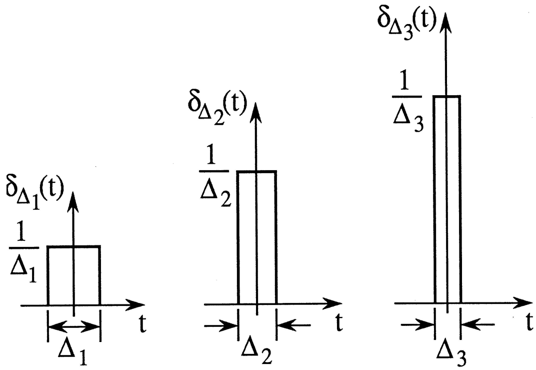
CT Impulse Function

Let $\delta_\Delta(t) = \frac{1}{\Delta}\text{rect}(\frac{t}{\Delta})$

Note that $\int_{-\infty}^{\infty} \delta_\Delta(t)dt = 1$.

What happens as $\Delta \to 0$?

$\Delta_1 > \Delta_2 > \Delta_3$



In the limit, we obtain

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

and

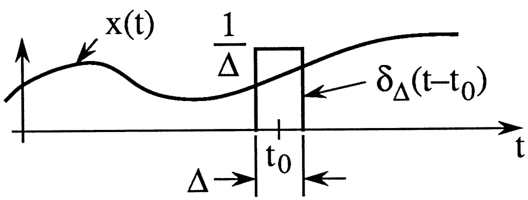
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

We will use impulses in three ways:

1. to sample signals
2. as a way to decompose signals into elementary components
3. as a way to characterize the response of a class of systems to an arbitrary input signal

Sifting property

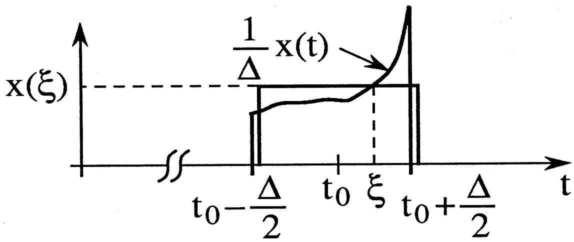
Consider a signal x(t) multiplied by $\delta_\Delta(t - t_0)$for some fixed t_0:



From the mean value theorem of calculus,

$$\int_{-\infty}^{\infty} x(t)\delta_\Delta(t-t_0)dt = x(\xi)$$

for some ξ which satisfies
$t_0 - \frac{\Delta}{2} \leq \xi \leq t_0 + \frac{\Delta}{2}$



As $\Delta \to 0, \xi \to t_0$ and $x(\xi) \to x(t_0)$
So we have
$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$
provided x(t) is continuous at t_0.

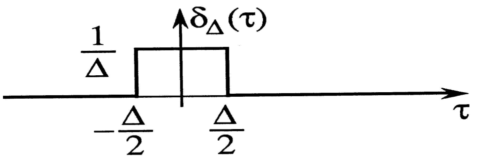
Equivalence

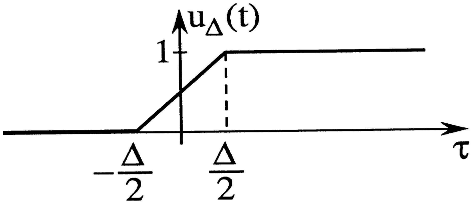
Based on sifting property,

$$x(t)\delta(t-t_0) \equiv x(t_0)\delta(t-t_0)$$

Indefinite integral of $\delta(t)$

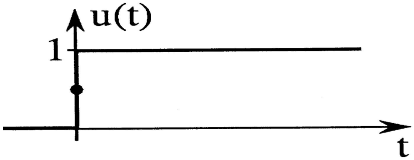
Let $u_\delta(t) = \int_{-\infty}^{t} \delta_\Delta(\tau)d\tau$





Let $\Delta \to 0$,

$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$

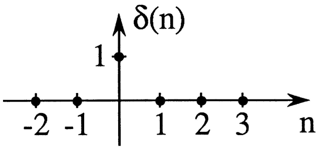


When we define the unit step function, we did not specify its value at t=0. In this case, $u(0) = 0.5$

More general form of sifting property

$$\int_a^b x(\tau)\delta(\tau - t_0)d\tau = \begin{cases} x(t_0), & a < t_0 < b \\ 0, & t_0 < a \text{ or } b < t_0 \end{cases}$$

DT Impulse Function (unit sample function)



$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases}$$

Sifting Property

$$\sum_{n=n_1}^{n_2} x(n)\delta(n-n_0) = \begin{cases} x(n_0), & n_1 \leq n_0 \leq n_2 \\ 0, & \text{else} \end{cases}$$

Equivalence

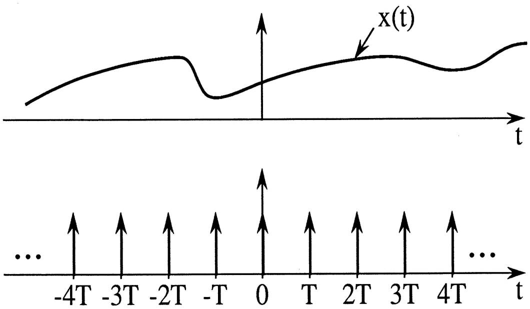
$$x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0)$$

Indefinite sum of $\delta(n)$

$$u(n) = \sum_{m=-\infty}^{n} \delta(m)$$

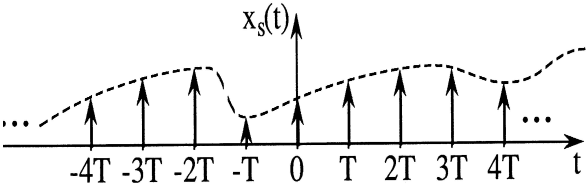
1.8 Comb and Replication Operations

Sifting property yields a single sample at t_0. Consider multiplying $x(t)$ by an entire train of impulses:

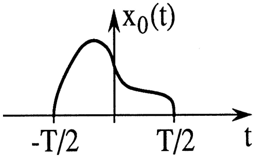


Define

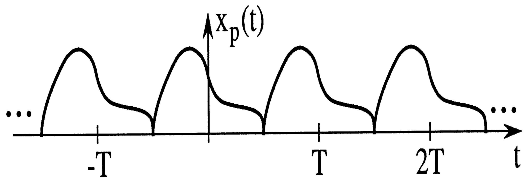
$$\begin{aligned} x_s(t) &= \text{comb}_T[x(t)] \\ &= x(t)\sum_n \delta(t-nT) \\ &= \sum_n x(t)\delta(t-nT) \\ &= \sum_n x(nT)\delta(t-nT) \end{aligned}$$



Area of each impulse = value of sample there.
The replication operator similarly provides a compact way to express periodic signals:



$$x_p(t) = \text{rep}_T[x_0(t)] = \sum_n x_0(t - nT)$$



Note that $x_0(t) = x_p(t)\text{rect}\left(\frac{t}{T}\right)$

Chapter 2

Systems

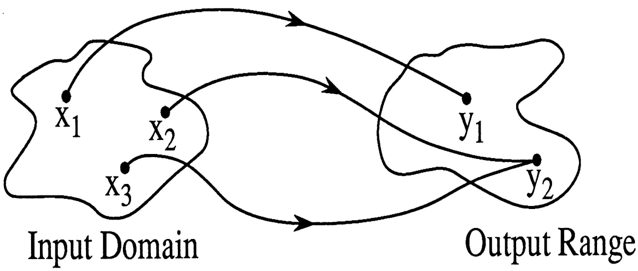
2.1 Chapter Outline

In this chapter, we will discuss:

1. System Properties
2. Convolution
3. Frequency Response

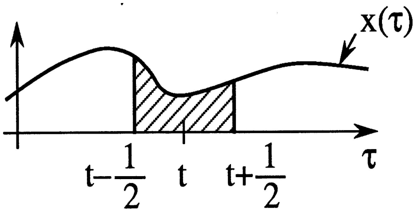
2.2 Systems

A system is a mapping which assigns to each signal $x(t)$ in the input domain a unique signal $y(t)$ in the output range.



Examples

1. CT $y(t) = \int_{t-1/2}^{t+1/2} x(\tau) d\tau$



let $x(t) = \sin(2\pi f t)$

$$y(t) = \int_{t-1/2}^{t+1/2} \sin(2\pi f t) dt$$
$$= \frac{-1}{2\pi f} \cos(2\pi f t)|_{t-1/2}^{t+1/2}$$
$$= -\frac{1}{2\pi f} \{\cos[2\pi f(t+1/2)] - \cos[2\pi f(t-1/2)]\}$$

use $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

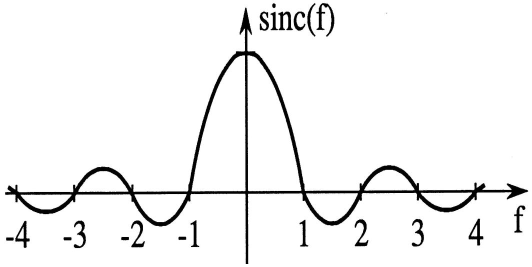
$$y(t) = -\frac{1}{2\pi f} \{[\cos(2\pi f t)\cos(\pi f) - \sin(2\pi f t)\sin(\pi f)]$$
$$-[\cos(2\pi f t)\cos(\pi f) + \sin(2\pi f t) sin(\pi f)]\}$$

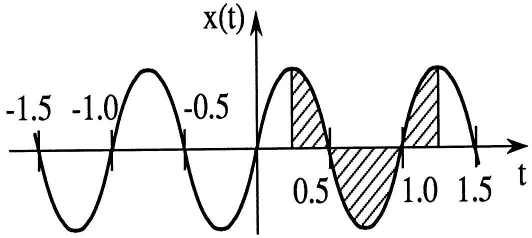
$$y(t) = \frac{sin(\pi f)}{\pi f} sin(2\pi f t)$$
$$= \text{sinc}(f) sin(2\pi f t)$$

Look at particular values of f

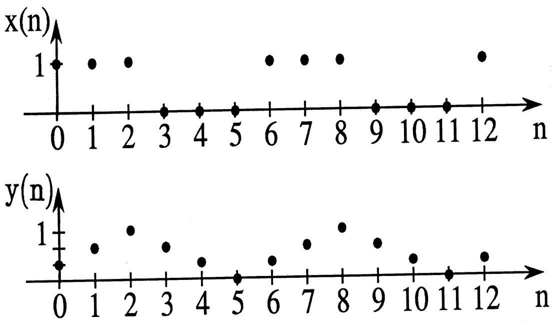
$f = 0: \ x(t) \equiv 0 \quad y(t) \equiv 0$

$f = 1: \ x(t) = \sin(2\pi t)$
$\text{sinc}(1) = 0$
$y(t) \equiv 0$





2. DT $y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$



Effect of filter on output:

1. widened,smoothed, or smeared each pulse
2. delayed pulse train by one sample time

2.3 System Properties

Notation:
$y(t) = S[x(t)]$

A. Linearity

def. A system S is *linear*(L) if for any two inputs $x_1(t)$ and $x_2(t)$ and any two constants a_1 and a_2, it satisfies:

$$S[a_1x_1(t) + a_2x_2(t)] = a_1S[x_1(t)] + a_2S[x_2(t)]$$

Special cases:

1. homogeneity (let $a_2 = 0$)
   $S[a_1x_1(t)] = a_1S[x_1(t)]$

2. superposition (let $a_1 = a_2 = 1$)
   $S[x_1(t) + x_2(t)] = S[x_1(t)] + S[x_2(t)]$

Examples

$y(t) = \int_{t-1/2}^{t+1/2} x(\tau)d\tau$
let $x_3(t) = a_1x_1(t) + a_2x_2(t)$

$$\begin{aligned} y_3(t) &= \int_{t-1/2}^{t+1/2} x_3(\tau)d\tau \\ &= \int_{t-1/2}^{t+1/2} [a_1x_1(\tau) + a_2x_2(\tau)]d\tau \\ &= a_1 \int_{t-1/2}^{t+1/2} x_1(\tau)d\tau + a_2 \int_{t-1/2}^{t+1/2} x_2(\tau)d\tau \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

$\therefore$ system is linear

We can similarly show that

$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$ is linear.

Consider two additional examples:

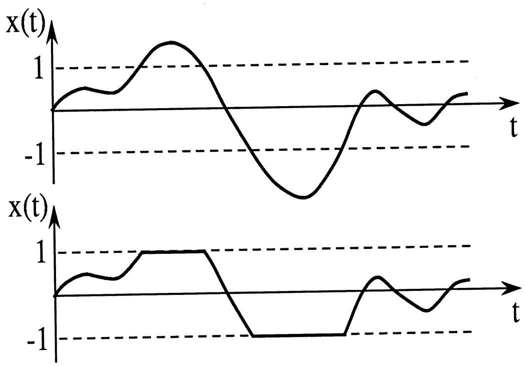
3. $y(n) = nx(n)$

let $x_3(n) = a_1x_1(n) + a_2x_2(n)$

$$\begin{aligned} y_3(n) &= nx_3(n) \\ &= (a_1nx_1(n) + a_2nx_2(n)) \\ &= a_1y_1(n) + a_2y_2(n) \end{aligned}$$

$\therefore$ system is linear

$$y(t) = \begin{cases} -1, & x(t) < -1 \\ x(t), & -1 \le x(t) \le 1 \\ 1, & 1 < x(t) \end{cases}$$



Suspect that system is nonlinear - find a counterexample.

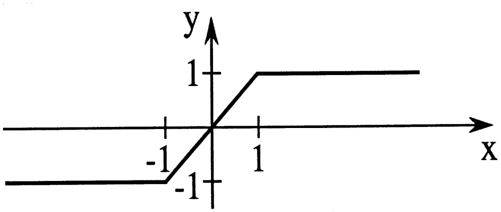
$x_1(t) \equiv 1 \Rightarrow y_1(t) \equiv 1$

$x_2(t) \equiv \frac{1}{2} \Rightarrow y_2(t) \equiv \frac{1}{2}$

$x_3(t) = x_1(t) + x_2(t) = \frac{3}{2} \;\Rightarrow\; y_3(t) \equiv 1 \neq \frac{3}{2}$

$\therefore$system is not linear

Because it is *memoryless*, this system is completely described by a curve relating input to output at each time t:



Importance of linearity: We can represent response to a complex input in terms of responses to very simple inputs.

B. Time Invariance

def. A system is *time-invariant* if delaying the input results in only an identical delay in the output, i.e.

$$\begin{aligned}
&\text{if } y_1(t) = S[x_1(t)] \\
&\text{and } y_2(t) = S[x_1(t - t_0)] \\
&\text{then } y_2(t) = y_1(t - t_0)
\end{aligned}$$

Examples

1. $y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$

assume

$y_1(n) = \frac{1}{3}[x_1(n) + x_1(n-1) + x_1(n-2)]$

let $x_2(n) = x_1(n - n_0)$

$$\begin{aligned}
y_2(n) &= \frac{1}{3}[x_2(n) + x_2(n-1) + x_3(n-2)] \\
&= \frac{1}{3}[x_1(n-n_0) + x_1(n-1-n_0) + x_1(n-2-n_0)] \\
&= \frac{1}{3}[x_1(n-n_0) + x_1(n-n_0-1) + x_1(n-n_0-2)] \\
&= y_1(n-n_0)
\end{aligned}$$

$\therefore$ system is TI.

We can similarly show that

2. $y(t) = \int_{t-1/2}^{t+1/2} x(\tau)d\tau$ is TI

3. $y(n) = nx(n)$

assume $y_1(n) = nx_1(n)$

let $x_2(n) = x_1(n - n_0)$

$$\begin{aligned} y_2(n) &= nx_2(n) \\ &= nx_1(n-n_0) \\ &\neq (n-n_0)x_1(n-n_0) \end{aligned}$$

$\therefore$ system is not TI.

C. Causality

def. A system S is *causal* if the output at time t depends only on $x(\tau)$ for $\tau \leq t$

Casuality is equivalent to the following property:

If $x_1(t) = x_2(t),\ \ t \leq t_0$ then $y_1(t) = y_2(t),\ \ t \leq t_0$

D. Stability

def. A system is said to be *bounded-input bounded-output (BIBO)* stable if every bounded input produces a bounded output,i.e.
$M_x < \infty \Rightarrow M_y < \infty$.

Example

1. $y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$

Assume $|x(n)| \leq M_x$ for all n

$$\begin{aligned} |y(n)| &= \frac{1}{3}|x(n) + x(n-1) + x(n-2)| \\ &\leq \frac{1}{3}[|x(n)| + |x(n-1)| + |x(n-2)|] \\ &\leq M_x \end{aligned}$$

2.4 Convolution

Characterization of behavior of an LTI system

1. Decompose input into sum of simple basis functions $x_i(n)$

$$x(n) = \sum_{i=0}^{N-1} c_i x_i(n)$$

$$\begin{aligned} y(n) &= S[x(n)] \\ &= \sum_{i=0}^{N-1} c_i S[x_i(n)] \end{aligned}$$

2. Choose basis functions which are all shifted versions of a single basis function $x_0(n)$

i.e. $x_i(n) = x_0(n - n_i)$

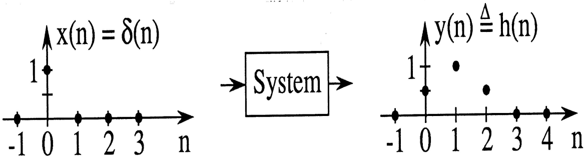
Let $y_i(n) = S[x_i(n)]$, i=0,...,N-1

then $y_i(n) = y_0(n - n_i)$

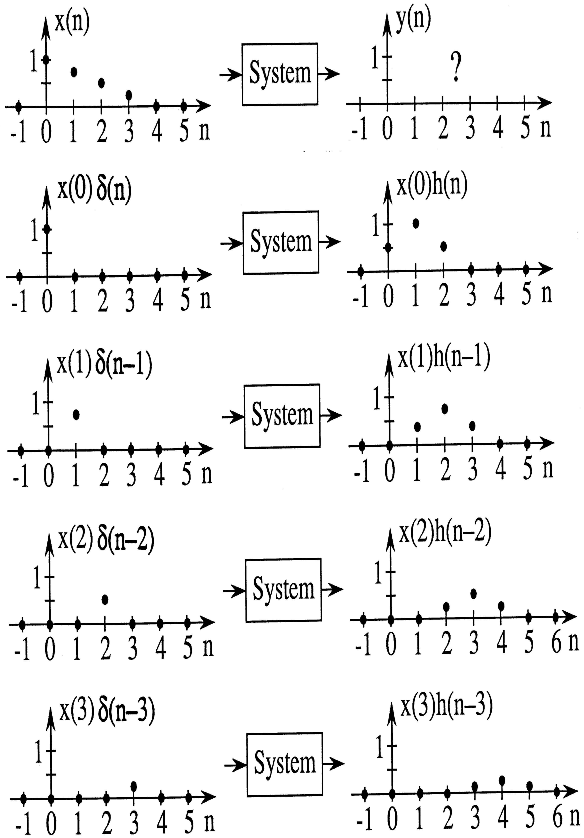
and $y(n) = \sum_{i=0}^{N-1} c_i y_0(n - n_i)$

3. Choose impulse as basis function

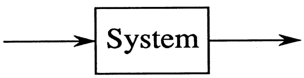
Denote impulse response by $h(n)$.



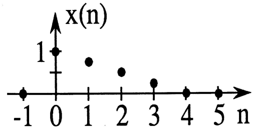
Now consider an arbitrary input $x(n)$.

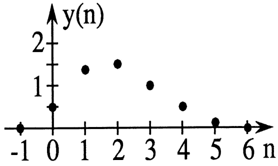


Sum over both the set of inputs and the set of outputs.



$$x(n) = x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3)$$
$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + x(3)h(n-3)$$





Convolution sum

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

let $\ell = n - k \Rightarrow k = n - \ell$

$$y(n) = \sum_{\ell=\infty}^{-\infty} x(n-\ell)h(\ell) = \sum_{\ell=-\infty}^{\infty} x(n-\ell)h(\ell)$$

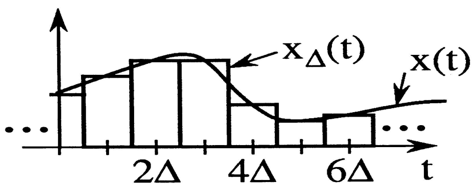
Notation and identity

For any signals $x_1(n)$ and $x_2(n)$, we use an asterisk to denote their convolution; and we have the following identity.

$$\begin{aligned} x_1(n) * x_2(n) &= \sum_{k=-\infty}^{\infty} x_1(n-k)x_2(k) \\ &= \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k) \end{aligned}$$

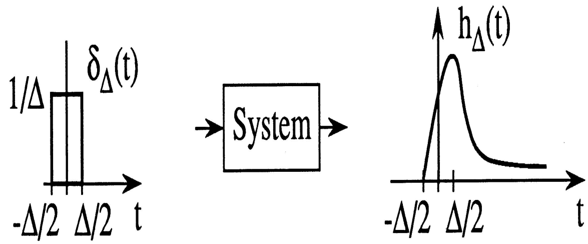
Characterization of CT LTI systems

- Approximate $x(t)$ by a superposition of rectangular pulses:



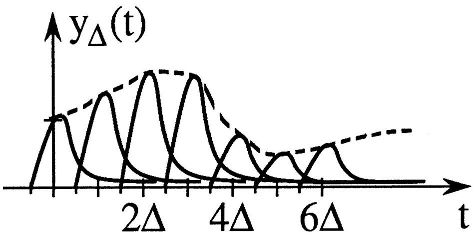
$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

- Find response to a single pulse:



- Determine response to $x_\Delta(t)$:

$$y_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t - k\Delta)\Delta$$



Let $\Delta \to 0$ $(d\tau)$ $k\Delta \to \tau$
$\delta_\Delta(t) \to \delta(t)$ $x_\Delta(t) \to x(t)$
$h_\Delta(t) \to h(t)$ $y_\Delta(t) \to y(t)$

Convolution Integral

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Notation and Identity

For any signals $x_1(t)$ and $x_2(t)$, we use an asterisk to denote their convolution; and we have the following identity.

$$\begin{aligned} x_1(n) * x_2(n) &= \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau \end{aligned}$$

<u>Example</u>

DT system $y(n) = \frac{1}{W}\sum_{k=0}^{W-1} x(n-k)$ W - integer

Find response to $x(n) = e^{-n/D}u(n)$.

W - width of averaging window

D - duration of input

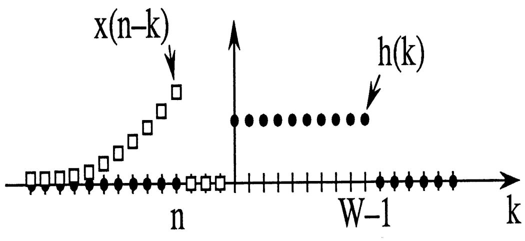
To find impulse response, let $x(n) = \delta(n) \Rightarrow h(n) = y(n)$.

$$h(n) = \frac{1}{W}\sum_{k=0}^{W-1} \delta(n-k) = \begin{cases} 1/W, & 0 \le n \le W-1 \\ 0, & \text{else} \end{cases}$$

Now use convolution to find response to $x(n) = e^{-n/D}u(n)$.

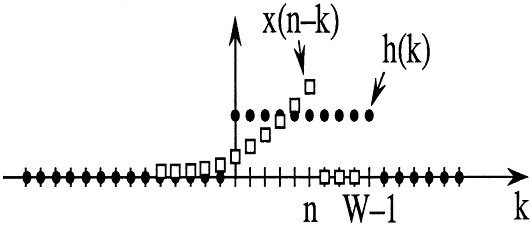
$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

Case 1: $n < 0$



$y(n) = 0$

Case 2: $0 \le n \le W-1$



$$\begin{aligned} y(n) &= \sum_{k=0}^{n} x(n-k)h(k) \\ &= \frac{1}{W}\sum_{k=0}^{n} e^{-(n-k)/D} \\ &= \frac{1}{W}e^{-n/D}\sum_{k=0}^{n} e^{k/D} \end{aligned}$$

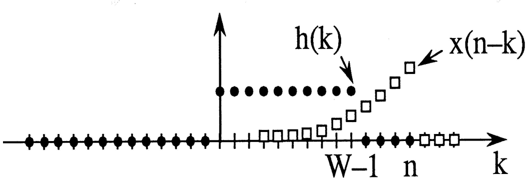
Geometrical Series

$$\sum_{k=0}^{N-1} z^k = \frac{1-z^N}{1-z}, \text{ for any complex number z}$$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \;\; |z|<1$$

$$y(n) = \frac{1}{W}e^{-n/D}\left[\frac{1-e^{(n+1)/D}}{1-e^{1/D}}\right]$$

Case 3: $W \le n$

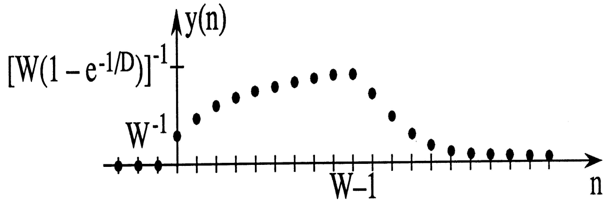


$$
\begin{aligned}
y(n) &= \sum_{k=0}^{W-1} x(n-k)h(k) \\
&= \frac{1}{W}\sum_{k=0}^{W-1} e^{-(n-k)/D} \\
&= \frac{1}{W}e^{-n/D}\sum_{k=0}^{W-1} e^{k/D}
\end{aligned}
$$

$$
\begin{aligned}
y(n) &= \frac{1}{W}e^{-n/D}\left[\frac{1-e^{W/D}}{1-e^{1/D}}\right] \\
&= \frac{1}{W}\left[\frac{1-e^{-(W/D}}{1-e^{-1/D}}\right]e^{-[n-(W-1)]/D}
\end{aligned}
$$

Putting everything together

$$
y(n) = \begin{cases} 0, & n < 0 \\ \frac{1}{W}\left[\frac{1-e^{-(n+1)/D}}{1-e^{-1/D}}\right], & 0 \le n \le W-1 \\ \frac{1}{W}\left[\frac{1-e^{-(W/D}}{1-e^{-1/D}}\right]e^{-[n-(W-1)]/D}, & W \le n \end{cases}
$$



Causality for LTI systems

$$
y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k) + \sum_{k=n+1}^{\infty} x(k)h(n-k)
$$

System will be causal $\Leftrightarrow$ second sum is zero for any input x(k).

This will be true $\Leftrightarrow$ $h(n-k) = 0,\ k = n+1, ..., \infty$

$\Leftrightarrow \ h(k) = 0, \ \ k < 0$

An LTI system is causal $\Leftrightarrow \ h(k) = 0, \, k < 0,$ i.e. the impulse response is a causal signal.

Stability of LTI systems

Suppose the input is bounded, i.e $M_x < \infty$

$y(n) = \sum_{k=\infty}^{\infty} x(k)h(n-k)$

$$\begin{aligned} |y(n)| &= |\sum_{k=-\infty}^{\infty} x(k)h(n-k)| \\ &\leq \sum_{k=-\infty}^{\infty} |x(k)||h(n-k)| \\ &\leq M_x \sum_{k=-\infty}^{\infty} |h(k)| \end{aligned}$$

Therefore, it is sufficient for BIBO stability that the impulse response be absolutely summable.

Suppose $\sum_k |h(k)| \not< \infty$.

Consider $y(0) = \sum_k x(k)h(-k)$.

Assuming h(k) to be real-valued, let $x(k) = \begin{cases} 1, & h(-k) > 0 \\ -1, & h(-k) < 0 \end{cases}$

then $y(0) = \sum_k |h(k)| \not< \infty$

For an LTI system to be BIBO stable, it is necessary that the impulse response is absolutely summable, i.e. $\sum_k |h(k)| < \infty$.

Examples

$y(n) = x(n) + y(n-1)$

Find the impulse response.

Let $x(n) = \delta(n)$, then $h(n) = y(n)$.

Need to find solution to

$y(n) = \delta(n) = 2y(n-1)$

This example differs from earlier ones because the system is recursive, i.e the current output depends on previous output values as well as the current and previous inputs.

1. must specify initial conditions for the system (assume y(-1)=0)
2. cannot directly write a closed form expression for y(n)

Find output sequence term by term.

$$y(0) = \delta(0) + 2y(-1) = 1 + 2(0) = 1$$

$$y(1) = \delta(1) + 2y(0) = 0 + 2(1) = 2$$

$$y(2) = \delta(2) + 2y(1) = 0 + 2(2) = 4$$

Recognize general form.
$h(n) = y(n) = 2^n u(n)$

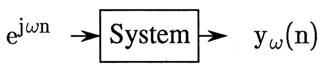
1. Assuming system is initially at rest, it is causal
2. $\sum_n |h(n)| \not< \infty \Rightarrow$ system is not BIBO stable

2.5 Frequency Response

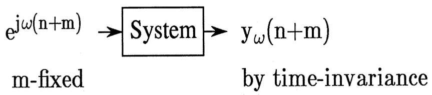
We have seen that the impulse signal provides a simple way to characterize the response of an LTI system to any input. Sinusoids play a similar role

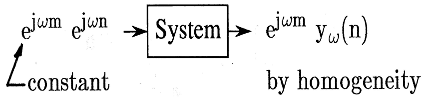
Consider $x(n) = e^{j\omega n}$, ω is fixed

Denote response by $y_\omega(n)$



Now consider





Since $e^{j\omega m}e^{j\omega n} = e^{j\omega(n+m)}$

$$y_\omega(n+m) = y_\omega(n)e^{j\omega m}$$

Let $n = 0$,

$$y_\omega(m) = y_\omega(0)e^{j\omega m} \text{ for all m}$$

$\therefore$ For a system which is homogeneous and time-invariant, the response to a complex exponential input $x(n) = e^{j\omega n}$ is $y(n) = y_\omega(0)x(n)$, a frequency-dependent constant times the input $x(n)$. Thus, complex exponential signals are eigenfunctions of homogeneous, time-invariant systems.

Comments

- We refer to the constant of proportionality as the frequency response of the system and denote it by

$$H(e^{j\omega}) = y_\omega(0)$$

- We write it as a function of $e^{j\omega}$ rather than ω for two reasons:

    1. digital frequencies are only unique modulo 2π
    2. there is a relation between frequency response and the Z transform that this representation captures

- Note that we did not use superposition. We will need it later when we express the response to arbitrary signals in terms of the frequency response.

Magnitude and Phase of Frequency Response

In general, $H(e^{j\omega})$ is complex-valued,
i.e. $H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$
Thus for $x(n) = e^{j\omega n}$
$y(n) = A(\omega)e^{j[\omega n + \angle\theta(\omega)]}$
Suppose response to any real-valued input $x(n)$ is real-valued.
Let $x(n) = \cos(\omega n) = \frac{1}{2}[e^{j\omega n} + e^{-j\omega n}]$.
Assuming superposition holds,

$$y(n) = \frac{1}{2}\left\{H(e^{j\omega})e^{j\omega n} + H(e^{-j\omega})e^{-j\omega n}\right\}$$

$$[y(n)]^* = \frac{1}{2}\left\{[H(e^{j\omega})]^* e^{-j\omega n} + [H(e^{-j\omega})]^* e^{j\omega n}\right\}$$

$$y(n) = [y(n)]^* \Leftrightarrow H(e^{-j\omega}) = [H(e^{j\omega}]^*$$

Expressed in polar coordinates
$A(-\omega)e^{j\angle\theta(-\omega)} = A(\omega)e^{-j\angle\theta(\omega)}$
$\therefore A(-\omega) = A(\omega)$ even
$\theta(-\omega) = -\theta(\omega)$ odd

Examples

1. $y(n) = \frac{1}{2}[x(n) + x(n-1)]$

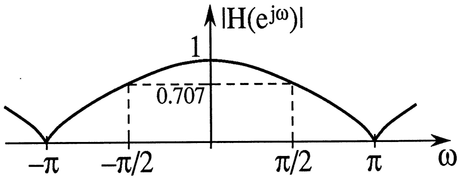
Let $x(n) = e^{j\omega n}$
$y(n) = \frac{1}{2}[e^{j\omega n} + e^{j\omega(n-1)}]$
$= \frac{1}{2}[1 + e^{-j\omega}]e^{j\omega n}$
$\therefore H(e^{j\omega}) = \frac{1}{2}[1 + e^{-j\omega}]$
Factoring out the half-angle:

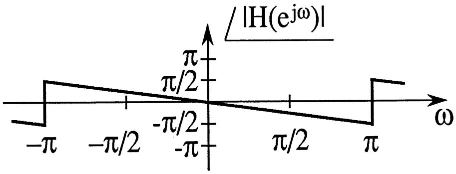
$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{2}e^{-j\omega/2}[e^{j\omega/2} + e^{-j\omega/2}] \\ &= e^{-j\omega/2}\cos(\omega/2) \end{aligned}$$

$$\begin{aligned}|H(e^{j\omega})| &= |e^{-j\omega/2}||\cos(\omega/2)| \\ &= |\cos(\omega/2)|\end{aligned}$$

$$\angle H(e^{j\omega}) = \angle e^{-j\omega/2} + \angle cos(\omega/2)$$

$$\angle H(e^{j\omega}) = \begin{cases} -\omega/2, & \cos(\omega/2) \geq 0 \\ -\omega/2 \pm \pi, & cos(\omega/2) < 0 \end{cases}$$





Note:

- even symmetry of $|H(e^{j\omega})|$
- odd symmetry of $\angle H(e^{j\omega})|$
- periodicity of $H(e^{j\omega})$ with period 2π
- low pass characteristic

2. $y(n) = \frac{1}{2}[x(n) - x(n-2)]$

Let $x(n) = e^{j\omega n}$

$$\begin{aligned}y(n) &= \frac{1}{2}[e^{j\omega n} - e^{j\omega(n-2)}] \\ &= \frac{1}{2}[1 - e^{-j\omega 2}]e^{j\omega n}\end{aligned}$$

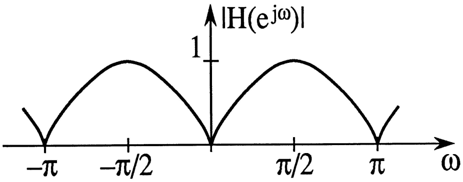
$$\begin{aligned}H(e^{j\omega}) &= \frac{1}{2}[1 - e^{-j\omega 2}] \\ &= je^{-j\omega}[\frac{1}{j2}(e^{j\omega} - e^{-j\omega})]\end{aligned}$$

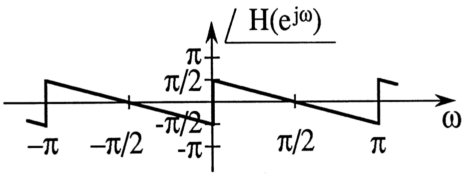
$H(e^{j\omega}) = je^{-j\omega}\sin(\omega)$

$$\begin{aligned} |H(e^{j\omega})| &= |j||e^{-j\omega}||\sin(\omega)| \\ &= |\sin(\omega)| \end{aligned}$$

$\angle H(e^{j\omega}) = \angle j + \angle e^{-j\omega} + \angle \sin(\omega)$

$$\angle H(e^{j\omega}) = \begin{cases} \pi/2 - \omega, & \sin(\omega) \geq 0 \\ \pi/2 - \omega \pm \pi, & \sin(\omega) < 0 \end{cases}$$





The filter has a bandpass characteristic.

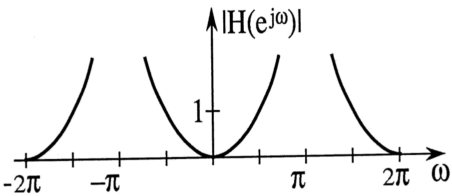
3. $y(n) = x(n) - x(n-1) - y(n-1)$
Let $x(n) = e^{j\omega n}$, how do we find $y(n)$?
Assume desired form of output,
i.e. $y(n) = H(e^{j\omega})e^{j\omega n}$
$H(e^{j\omega})e^{j\omega n} = e^{j\omega n} - e^{j\omega(n-1)} - H(e^{j\omega})e^{j\omega(n-1)}$

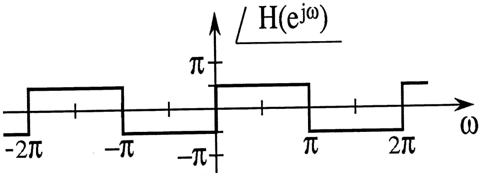
$$H(e^{j\omega})[1 + e^{-j\omega}]e^{j\omega n} = [1 - e^{-j\omega}]e^{j\omega n}$$

$$\begin{aligned} H(e^{j\omega}) &= \left[\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right] \\ &= \frac{je^{-j\omega/2}[\frac{1}{j2}(e^{j\omega/2} - e^{-j\omega/2})]}{e^{-j\omega/2}[\frac{1}{2}(e^{j\omega/2} + e^{-j\omega/2})]} \\ &= j\frac{\sin(\omega/2)}{\cos(\omega/2)} \\ &= j\tan(\omega/2) \end{aligned}$$

$$|H(e^{j\omega})| = |\tan(\omega/2)|$$

$$\angle H(e^{j\omega}) = \begin{cases} \pi/2, & \tan(\omega/2) \geq 0 \\ \pi/2 \pm \pi, & \tan(\omega/2) < 0 \end{cases}$$





Comments

- What happens at $\omega = \pi/2$?
- Factoring out the half-angle is possible only for a relatively restricted class of filters

Chapter 3

Fourier Analysis

3.1 Chapter Outline

In this chapter, we will discuss:

1. Continuous-Time Fourier Series (CTFS)
2. Continuous-Time Fourier Transform (CTFT)
3. Discrete-Time Fourier Transorm (DTFT)

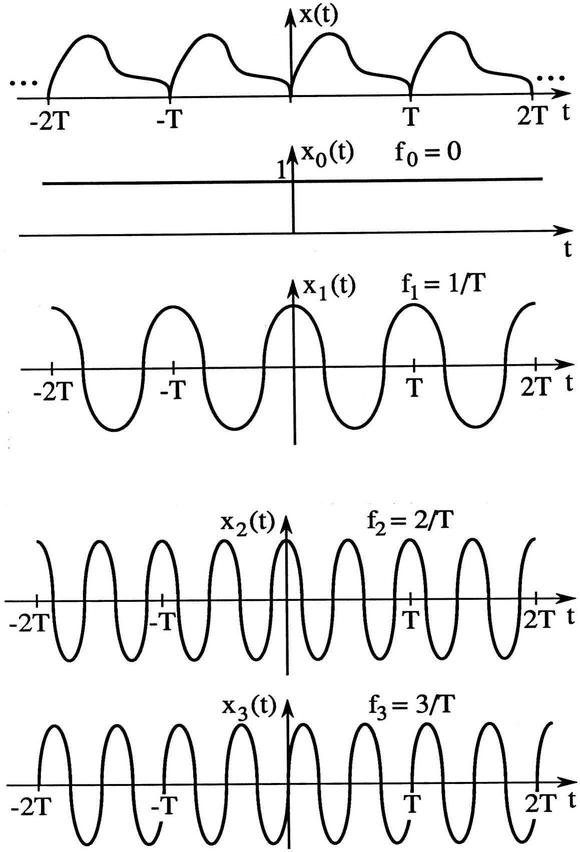
3.2 Continuous-Time Fourier Series (CTFS)

Spectral representation for periodic CT signals
Assume $x(t) = x(t + nT)$
We seek a representation for x(t) of the form

$$x(t) = \frac{1}{T}\sum_{k=0}^{N-1} A_k x_k(t) \text{ (1a)}$$

$$x_k(t) = \cos(2\pi f_k t + \theta_k) \text{ (1b)}$$

Since $x(t)$ is periodic with period T, we only use the frequency components that are periodic with this period:



How do we choose amplitude A_k and the phase θ_k of each component?
$k \neq 0$:

$$
\begin{aligned}
A_k x_k(t) &= A_k \cos(2\pi kt/T + \theta_k) \\
&= \frac{A_k}{2} e^{j\theta_k} e^{j2\pi kt/T} + \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi kt/T} \\
&= X_k e^{j2\pi kt/T} + X_{-k} e^{-j2\pi kt/T}
\end{aligned}
$$

$k = 0$:

$$
\begin{aligned}
A_0 x_0(t) &= A_0 \cos(\theta_0) \\
&= X_0
\end{aligned}
$$

Given a fixed signal $x(t)$, we don't know at the outset whether an exact representation of the form given by Eq.(1) even exists.

However, we can always approximate $x(t)$ by an expression of this form. So we consider

$$\widehat{x}(t) = \frac{1}{T} \sum_{k=-(N-1)}^{N-1} X_k e^{j2\pi kt/T}$$

We want to choose the coefficients X_k so that $\widehat{x}(t)$ is a good approximation to $x(t)$.

Define $e(t) = \widehat{x}(t) - x(t)$.

Recall

$$\begin{aligned}
P_e &= \frac{1}{T} \int_{-T/2}^{T/2} |e(t)|^2 dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} |\widetilde{x}(t) - x(t)|^2 dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} \left| \frac{1}{T} \sum_{k=-(N-1)}^{N-1} X_k e^{j2\pi kt/T} - x(t) \right|^2 dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} \left[\frac{1}{T} \sum_{k=-(N-1)}^{N-1} X_k e^{j2/pikt/T} - x(t) \right] \\
&\times \left[\frac{1}{T} \sum_{\ell=-(N-1)}^{N-1} X_\ell^* e^{-j2\pi\ell t/T} - x^*(t) \right] dt
\end{aligned}$$

$$\begin{aligned}
P_e &= \frac{1}{T} \int_{-T/2}^{T/2} \left\{ \left[\frac{1}{T} \sum_{k=-N+1}^{N-1} X_k e^{j2\pi kt/T} \right] \right. \\
&- \left[\frac{1}{T} \sum_{\ell=-N+1}^{N-1} X_\ell^* e^{-j2\pi\ell t/T} \right] \\
&- \left[\frac{1}{T} \sum_{k=-N+1}^{N-1} X_k e^{j2/pikt/T} \right] x^*(t) - \\
&x(t) \left. \left[\frac{1}{T} \sum_{\ell=-N+1} X_\ell^* e^{-j2\pi\ell t/T} \right] + x(t)x^*(t) \right\} dt
\end{aligned}$$

$$\begin{aligned} P_e = & \frac{1}{T^3} \sum_{k=-N+1}^{N-1} \sum_{\ell=-N+1}^{N-1} X_k X_\ell^* \int_{-T/2}^{T/2} e^{j2\pi(k-l)t/T} dt \\ & - \frac{1}{T^2} \sum_{k=-N+1}^{N-1} X_k \int_{-T/2}^{T/2} x^*(t) e^{j2\pi kt/T} dt \\ & - \frac{1}{T^2} \sum_{\ell=-N+1}^{N-1} X_\ell^* \int_{-T/2}^{T/2} x(t) e^{-j2\pi lt/T} dt \\ & + \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \end{aligned}$$

$$\begin{aligned} \int_{-T/2}^{T/2} e^{j2\pi(k-l)t/T} dt = & \frac{T}{j2\pi(k-l)} e^{j2\pi(k-l)t/T} \Big|_{-T/2}^{T/2} \\ = & \frac{T}{j2\pi(k-\ell)} \left[e^{[j\pi(k-\ell)]} - e^{[-j\pi(k-\ell)]} \right] \\ = & T \text{sinc}(k-\ell) \\ = & \begin{cases} T, & k = l \\ 0, & k \neq l \end{cases} \end{aligned}$$

Let $\widetilde{X}_l = \int_{-T/2}^{T/2} x(t) e^{-j2\pi lt/T} dt.$

$$P_e = \frac{1}{T^2} \sum_{-N+1}^{N-1} \left\{ |x(k)|^2 - X_k \widetilde{X}_k^* - X_k^* \widetilde{X}_k \right\} + P_x$$

X_k-unknown coefficients in the Fourier Series approximation

$\widetilde{X}_k$-fixed numbers that depend on $x(t)$

We want to choose values for the coefficients X_k,$k = -N+1, ..., N-1$ which will minimize P_e.

Fix l between $-N+1$ and $N-1$.

Let $X_l = A_l + jB_l$.

and consider $\frac{\partial P_e}{\partial A_l}$ and $\frac{\partial P_e}{\partial B_l}$

$$P_e = \frac{1}{T^2} \sum_{k=-N+1}^{N-1} \left\{ |X(k)|^2 X_k \widetilde{X}_k^* - X_k^* \widetilde{X}_k \right\} + P_x$$

$$\begin{aligned} \frac{\partial P_e}{\partial A_l} &= \frac{1}{T^2} \frac{\partial P_e}{\partial A_l} \left\{ (A_l + jB_l)(A_l - jB_l) - (A_l + jB_l)\widetilde{X}_l^* - (A_l - jB_l)\widetilde{X}_l \right\} \\ &= \frac{1}{T^2} \left\{ X_l^* + X_l - \widetilde{X}_l^* - \widetilde{X}_l \right\} \\ &= \frac{2}{T^2} Re \left\{ X - l - \widetilde{X}_l \right\} \end{aligned}$$

$$\frac{\partial P_e}{\partial A_l} = 0 \;\Rightarrow\; Re\{X_l\} = Re\left\{\widetilde{X}_l\right\}$$

Similarly

$$\begin{aligned} \frac{\partial P_e}{\partial B_l} &= \frac{1}{T^2} jX_l^* - jX_l - j\widetilde{X}_l^* + j\widetilde{X}_l \\ &= j\frac{2}{T^2} Im \left\{ \widetilde{X}_l - X_l \right\} \end{aligned}$$

and

$$\frac{\partial P_e}{\partial B_l} = 0 \;\Rightarrow\; Im\{X_l\} = Im\left\{\widetilde{X}_l\right\}$$

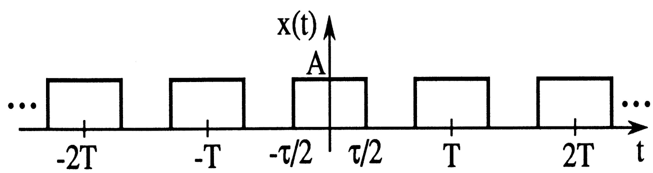
To summarize

For $\widehat{x}(t) = \frac{1}{T} \sum_{k=-N+1}^{N-1} X_k e^{j2\pi kt/T}$

to be a minimum mean-squared error approximation to the signal $x(t)$ over the interval $-T/2 \le t \le T/2$, the coefficients X_k must satisfy

$$X_k = \int_{-T/2}^{T/2} x(t) e^{-j2\pi kt/T} dt$$

Example

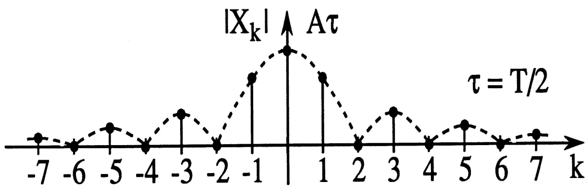


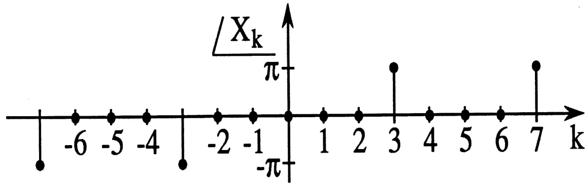
$$
\begin{aligned}
X_k &= \int_{-T/2}^{T/2} x(t) e^{-j2\pi kt/T} dt \\
&= A \int_{-\tau/2}^{\tau/2} e^{-j2\pi kt/T} dt \\
&= \frac{A}{-j2\pi kt/T} e^{-j2\pi kt/T} \\
&= \frac{A}{-j2\pi k/T} [e^{-j\pi k\tau/T} - e^{j\pi k\tau/T}] \\
&= A_\tau \frac{\sin(\pi k\tau/T)}{\pi k\tau/T}
\end{aligned}
$$

Line spectrum

$X_k = A_\tau \text{sinc}(k\tau/T)$

$|X_k| = |A_\tau \text{sinc}(k\tau/T)| \quad \angle X_k = \begin{cases} 0, & \text{sinc}(k\tau/T) > 0 \\ \pm\pi, & \text{sinc}(k\tau/T) < 0 \end{cases}$

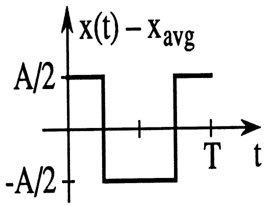




Comments

Recall $X_k = \int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt$

1. $X_0 = \int_{-T/2}^{T/2} x(t)dt = Tx_{\text{avg}}$
2. $X_k = 0, k \neq 0$
3. $\lim_{T\to\infty} |X_k| = 0$



What happens as N increases?

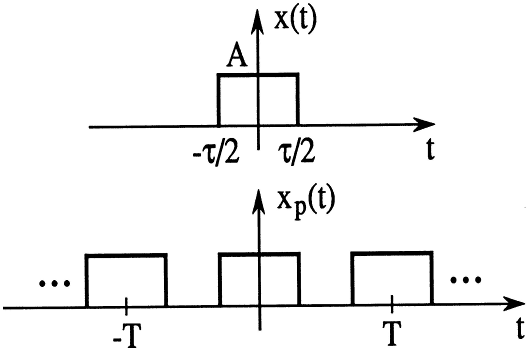
Let $\widehat{x}_N(t) = \frac{1}{T}\sum_{k=-N+1}^{N-1} X_k e^{j2\pi kt/T}$

1. If $\int_{-T/2}^{T/2} |x(t)|^2 dt < \infty$, $\int_{-T/2}^{T/2} |\widehat{x}_N(t) - x(t)|^2 dt \to 0$
2. If $\int_{-T/2}^{T/2} |x(t)|dt < \infty$ and other Dirichilet conditions are met, $\widehat{x}_N(t) \to x(t)$ for all t where $x(t)$ is continuous
3. In the neighborhood of discontinuities, $\widehat{x}_N(t)$ exhibits an overshoot or undershoot with maximum amplitude equal to 9 percent of the step size no matter how large N is (Gibbs phenomena)

3.3 Continuous-Time Fourier Transform

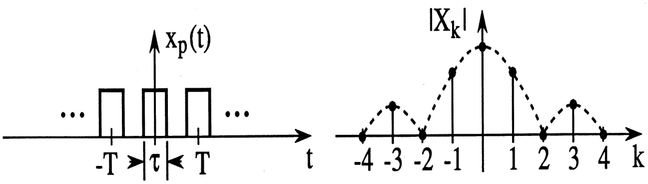
Spectral representation for aperiodic CT signals

Consider a fixed signal $x(t)$ and let $x_p(t) = \text{rep}_T[x(t)]$.

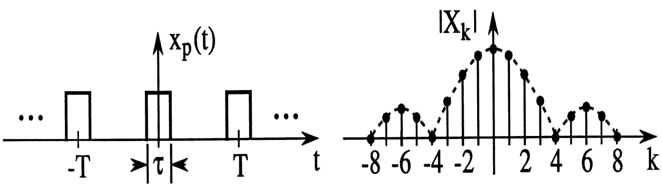


What happens to the Fourier series as T increases?

$\tau = T/2$



$\tau = T/4$



Fourier coefficients

$$X_k = \int_{-T/2}^{T/2} x(t) e^{-j2\pi kt/T} dt$$

Let $T \to \infty$.

$k/T \to f$

$X_k \to X(f)$

$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

Fourier series expansion

$x_p(t) = \frac{1}{T}\sum_{-\infty}^{\infty} X_k e^{j2\pi kt/T}$

Let $T \to \infty$.

$x_p(t) \to x(t)$

$k/T \to f \quad X_k \to X(f)$

$\frac{1}{T}\sum_{k=-\infty}^{\infty} \to \int_{-\infty}^{\infty} df$

$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$

Fourier Transform pairs

Forward transform

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Inverse Transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Sufficient conditions for Existence of CTFT

1. $x(t)$ has finite energy
   $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$
2. $x(t)$ is absolutely integrable
   $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
   and it satisfies Dirichlet conditions

Transform relations

- Linearity

  $a_1 x_1(t) + a_2 x_2(t) \overset{CTFT}{\leftrightarrow} a_1 X_1(f) + a_2 X_2(f)$

- Scaling and shifting

  $x\left(\frac{t-t_0}{a}\right) \overset{CTFT}{\leftrightarrow} |a| X(af) e^{-j2\pi f t_0}$

- Modulation

  $x(t) e^{j2\pi f_0 t} \overset{CTFT}{\leftrightarrow} X(f - f_0)$

- Reciprocity

  $X(t) \overset{CTFT}{\leftrightarrow} x(-f)$

- Parseval's relation

  $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$

- Initial value

  $\int_{-\infty}^{\infty} x(t) dt = X(0)$

Comments

1. Reflection is a special case of scaling and shifting with $a = -1$ and $t_0 = 0$, i.e.
   $x(-t) \overset{CTFT}{\leftrightarrow} X(-f)$

2. The scaling relation exhibits reciprocal spreading

3. Uniqueness of the CTFT follows from Parseval's relation

CTFT for real signals

If x(t) is real, $X(f) = [X(-f)]^*$.
$\Rightarrow |X(f)| = |X(-f)|$ and $\angle X_f = -\angle X(-f)$
In this case, the inverse transform may be written as:

$$x(t) = 2\int_0^{\infty} |X(f)| \cos[2\pi f t + \angle X(f)] df$$

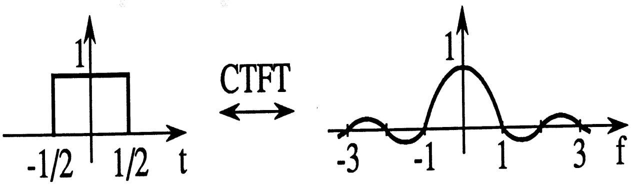
Additional symmetry relations:

$x(t)$ is real and even $\Leftrightarrow$ $X(f)$ is real and even.

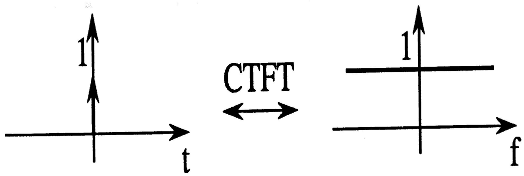
$x(t)$ is real and odd $\Leftrightarrow$ $X(f)$ is real and odd.

Important transform pairs:

- $\text{rect}(t) \overset{CTFT}{\leftrightarrow} \text{sinc}(f)$



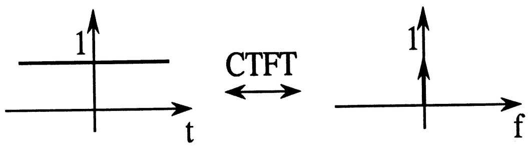
- $\delta(t) \overset{CTFT}{\leftrightarrow} 1$ (by sifting property)



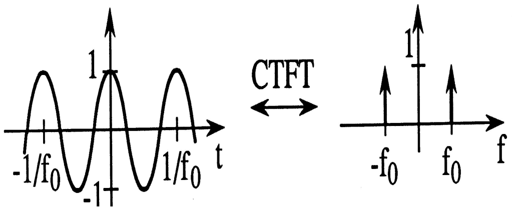
Proof

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = 1$$

- $1 \overset{CTFT}{\leftrightarrow} \delta(f)$ (by reciprocity)



- $e^{j2\pi f_0 t} \overset{CTFT}{\leftrightarrow} \delta(f - f_0)$ (by modulation property)



- $\cos(2\pi f_0 t) \overset{CTFT}{\leftrightarrow} \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$

Generalized Fourier transform

Note that $\delta(t)$ is absolutely integrable but not square integrable.

Consider $\delta_\Delta(t) = \frac{1}{T}\text{rect}\left(\frac{t}{\Delta}\right)$.

$$\int_{-\infty}^{\infty} |\delta_\Delta(t)|dt = 1$$

$$\int_{-\infty}^{\infty} |\delta_\Delta(t)|^2 dt = \frac{1}{\Delta}$$

$$\therefore \lim_{\Delta\to 0}\int_{-\infty}^{\infty} |\delta_\Delta(t)|^2 dt = \infty$$

The function $x(t) \equiv 1$ is neither absolutely nor square integrable; and the integral

$\int_{-\infty}^{\infty} 1e^{-j2\pi ft}dt$ is undefined.

Even when neither condition for existence of the CTFT is satisfied, we may still be able to define a Fourier transform through a limiting process.

Let $x_n(t)$, $n = 0, 1, 2, ...$ denote a sequence of functions each of which has a valid CTFT $X_n(f)$.

Suppose $\lim_{n\to\infty} x_n(t) = x(t)$, that function does not have a valid transform.

If $X(f) = \lim_{n\to\infty} X_n(f)$ exists, we call it the *generalized* Fourier transform of $x(t)$ i.e.

$$x_0(t) \overset{CTFT}{\leftrightarrow} X_0(f)$$

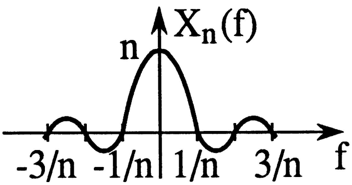
$$x_1(t) \overset{CTFT}{\leftrightarrow} X_1(f)$$

$$x_2(t) \overset{CTFT}{\leftrightarrow} X_2(f)$$

$$x(t) \overset{GCTFT}{\leftrightarrow} X(f)$$

Example

Let $x_n(t) = \text{rect}(t/n)$.
$X_n(f) = n\text{sinc}(nf)$ (by scaling)
$\lim_{n\to\infty} x_n(t) = 1$



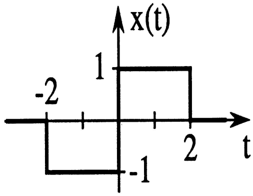
What is $\lim_{n\to\infty} X_n(f)$?
$X_n(f) \to 0, \quad f \neq 0$
$X_n(0) \to \infty$
What is $\int_{-\infty}^{\infty} X_n(f)df$?
By the initial value relation

$$\int_{-\infty}^{\infty} X_n(f)df = x_n(0) = 1$$

$\therefore \lim_{n\to\infty} X_n(f) = \delta(f)$
and we have
$1 \overset{GCTFT}{\leftrightarrow} \delta(f)$

Efficient calculation of Fourier transforms

Suppose we wish to determine the CTFT of the following signal.



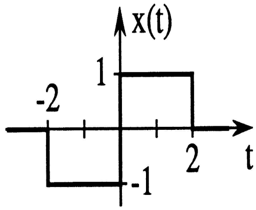
Brute force approach:

1. Evaluate transform integral directly
   $X(f) = \int_{-2}^{0} (-1)e^{-j2\pi ft}dt + \int_{0}^{2} (1)e^{-j2\pi ft}dt$

2. Collect terms and simplify

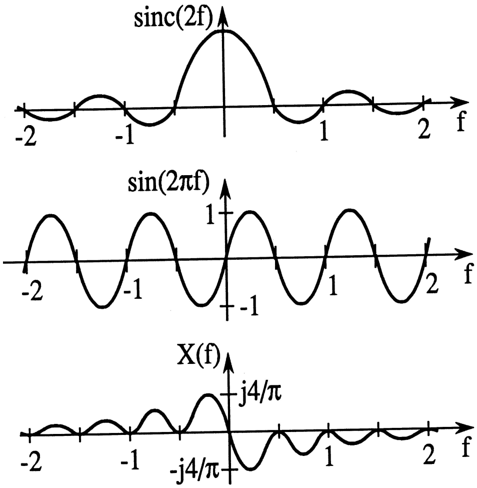
Faster approach

1. Write $x(t)$ in terms of functions whose transforms are known



$$x(t) = -\text{rect}\left(\frac{t+1}{2}\right) + \text{rect}\left(\frac{t-1}{2}\right)$$

2. Use transform relations to determine X_f
   $X(f) = 2\ \text{sinc}(2f)[e^{-j2\pi f} - e^{j2\pi f}]$
   $X(f) = -j4\ \text{sinc}(2f)sin(2\pi f)$



Comments

1. $A_x = 0$ and $X(0) = 0$
2. $x(t)$ is real and odd and $X(f)$ is imaginary and odd

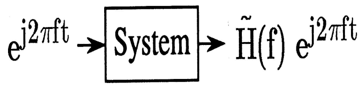
CTFT and CT LTI systems

The key factor that made it possible to express the response $y(t)$ of an LTI system to an *arbitrary* input $x(t)$ in terms of the impulse response $h(t)$ was the fact that we could write $x(t)$ as the superposition of impulses $\delta(t)$.

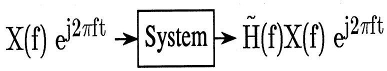
We can similarly express $x(t)$ as a superposition of complex exponential signals:

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

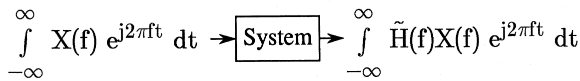
Let $\widetilde{H}(f)$ denote the frequency response of the system, i.e. for a fixed frequency f.



then by homogeneity



and by superposition



Thus, the response to $x(t)$ is

$$y(t) = \int_{-\infty}^{\infty} \widetilde{H}(f)X(f)e^{j2\pi ft}dt$$

But also

$$y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}dt$$

$\therefore Y(f) = \widetilde{H}(f)X(f)$ (1)
We also know that
$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$

What is the relation between $h(t)$ and $\widetilde{H}(f)$?

Let $x(t) = \delta(t) \;\Rightarrow\; y(t) = h(t)$

then $X(f) = 1$ and $Y(f) = H(f)$.

From Eq(1), we conclude that $\widetilde{H}(f) = H(f)$

Since the frequency response is the CTFT of the impulse response, we will drop the tilde.

Summarizing, we have two equivalent characterizations for CT LTI systems.

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$Y(f) = X(f)H(f)$$

Convolution Theorem

Since $x(t)$ and $h(t)$ are arbitrary signals, we also have the following Fourier transform relation

$$\int x_1(\tau)x_2(t-\tau)d\tau \overset{CTFT}{\leftrightarrow} X_1(f)X_2(f)$$

$$x_1(t) * x_2(t) \overset{CTFT}{\leftrightarrow} X_1(f)X_2(f)$$

Product Theorem

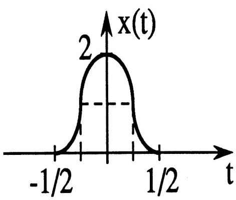
By reciprocity, we also have the following result:

$$x_1(t)x_2(t) \overset{CTFT}{\leftrightarrow} X_1(f) * X_2(f)$$

This can be very useful for calculating transforms of certain functions.

Example $x(t) = \begin{cases} \frac{1}{2}[1+\cos(2\pi t)], & |t| \leq 1/2 \\ 0, & |t| > 1/2 \end{cases}$

Find $X(f)$



$$x(t) = \frac{1}{2}[1+\cos(2\pi t)]\text{rect}(t)$$

$$\therefore X(f) = \frac{1}{2}\left\{\delta(f) + \frac{1}{2}[\delta(f-1)+\delta(f+1)]\right\} * \text{sinc}(f)$$

Since convolution obeys linearity, we can write this as

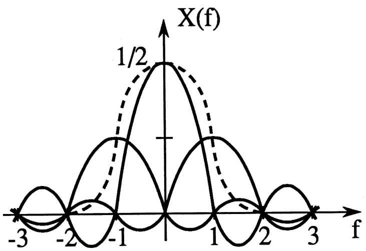
$$X(f) = \frac{1}{2}\left\{\delta(f) * \text{sinc}(f) + \frac{1}{2}[\delta(f-1) * \text{sinc}(f) + \delta(f+1) * \text{sinc}(f)]\right\}$$

All three convolutions here are of the same general form.

Identity

For any signal $w(t)$,
$w(t) * \delta(t - t_0) = w(t - t_0)$
Proof:
$w(t) * \delta(t - t_0) = \int w(\tau)\delta(t - \tau - t_0)d\tau = w(t - t_0)$ (by sifting property)
Using the identity,

$$\begin{aligned} X(f) &= \frac{1}{2}\left\{\delta(f) * \text{sinc}(f) + \frac{1}{2}[\delta(f-1) * \text{sinc}(f) + \delta(f+1) * \text{sinc}(f)]\right\} \\ &= \frac{1}{2}\left\{\text{sinc}(f) + \frac{1}{2}[\text{sinc}(f-1) + \text{sinc}(f+1)]\right\} \end{aligned}$$



Fourier transform of Periodic Signals

- We previously developed the Fourier series as a spectral representation for periodic CT signals
- Such signals are neither square integrable nor absolutely integrable, and hence do not satisfy the conditions for existence of the CTFT
- By applying the concept of the generalized Fourier transform, we can obtain a Fourier transform for periodic signals

- This allows us to treat the spectral analysis of all CT signals within a single framework
- We can also obtain the same result directly from the Fourier series

Let $x_0(t)$ denote one period of a signal that is periodic with period T, i.e. $x_0(t) = 0, \quad |t| > T/2$.

Define $x(t) = \text{rep}_T[x_0(t)]$.

The fourier series representation for $x(t)$ is

$x(t) = \frac{1}{T}\sum_k X_k e^{j2\pi kt/T}$

Taking the CTFT directly, we obtain

$$\begin{aligned} X(f) &= F\left\{\frac{1}{T}\sum_k X_k e^{j2\pi kt/T}\right\} \\ &= \frac{1}{T}\sum_k X_k F\left\{e^{j2\pi kt/T}\right\} \text{ (by linearity)} \\ &= \frac{1}{T}\sum_k X_k \delta(f - k/T) \end{aligned}$$

Also

$$\begin{aligned} X_k &= \int_{-T/2}^{T/2} x(t)e^{-j2\pi kt/T}dt \\ &= \int_{-\infty}^{\infty} x_0(t)e^{-j2\pi kt/T}dt \\ &= X_0(k/T) \end{aligned}$$

Thus

$X(f) = \frac{1}{T}\sum_k X_0(k/T)\delta(f - k/T)$

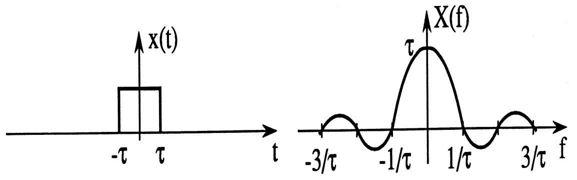
$= \frac{1}{T}\text{comb}_{\frac{1}{T}}[X_0(f)]$

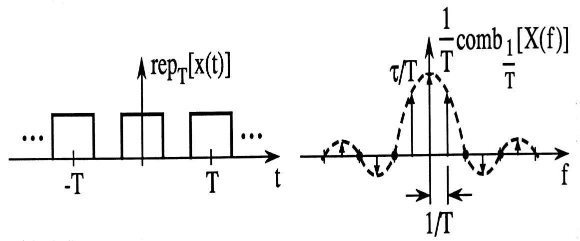
Dropping the subscript 0, we may state this result in the form of a transform relation:

$$\text{rep}_T[x(t)] \overset{CTFT}{\leftrightarrow} \frac{1}{T}\text{comb}_{\frac{1}{T}}[X(f)]$$

For our derivation, we required that $x(t) = 0$, $|t| > t/2$. However, when the generalized transform is used to derive the result, this restriction is not needed.

Example





3.4 Discrete-Time Fourier Transform

- Spectral representation for *aperiodic* DT signals
- As in the CT case, we may derive the DTFT by starting with a spectral representation (the discrete-time Fourier series) for periodic DT signals and letting the period become infinitely long
- Instead, we will take a shorter but less direct approach

Recall the continuous-time fourier series:

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T} \quad (1)$$

$$X_k = \int_{-T/2}^{T/2} x(t) e^{-j2\pi kt/T} dt \quad (2)$$

Before, we interpreted (1) as a spectral representation for the CT periodic signal x(t).

Now let's consider (2) to be a spectral representation for the sequence $X_k, -\infty < k < \infty$.

We are effectively interchanging the time and frequency domains.

We want to express an arbitrary signal x(n) in terms of complex exponential signals of the form $e^{j\omega n}$.

Recall that ω is only unique modulo 2π.

To obtain this, we make the following substitutions in (2).

$$X_k = \int_{-T/2}^{T/2} x(t) e^{-j2\pi kt/T} dt$$

$k \to n$

$Tx(t) \to X(e^{j\omega})$

$-2\pi t/T \to \omega$

$X_k \to x(n)$

dt$\to -\frac{T}{2\pi} d\omega$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

This is the inverse transform.

To obtain the forward transform, we make the same substitution in (1).

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}$$

$Tx(t) \to X(e^{j\omega})$

$k \to n$

$X_k \to x(n)$

$2\pi t/T \to -\omega$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Putting everything together

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Sufficient conditions for existence
$\sum_n |x(n)|^2 < \infty$ or

$\sum_n |x(n)| < \infty$ plus Dirichlet conditions.

Transform Relations

1. linearity

$$a_1x_1(n) + a_2x_2(n) \overset{DTFT}{\leftrightarrow} a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})$$

2. shifting

$$x(n - n_0) \overset{DTFT}{\leftrightarrow} X(e^{j\omega})e^{-j\omega n_0}$$

3. modulation

$$x(n)e^{j\omega_0 n} \overset{DTFT}{\leftrightarrow} X(e^{j(\omega-\omega_0)})$$

4. Parseval's relation

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

5. Initial value

$$\sum_{n=-\infty}^{\infty} x(n) = X(e^{j0})$$

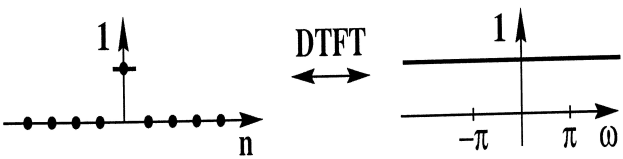
Comments

1. As discussed earlier, scaling in DT involves sampling rate changes; thus, the transform relations are more complicated than in CT case
2. There is no reciprocity relation, because time domain is a discrete-parameter whereas frequency domain is a continuous-parameter
3. As in CT case, Parseval's relation guarantees uniqueness of the DTFT

Some transform pairs

1. $x(n) = \delta(n)$

$$\begin{aligned} X(e^{j\omega}) &= \sum_n \delta(n) e^{-j\omega n} \\ &= 1 \text{ (by sifting property)} \end{aligned}$$



2. $x(n) = 1$
   - does not satisfy existence conditions
   - $\sum_n e^{-j\omega n}$ does not converge in ordinary sense
   - cannot use reciprocity as we did for CT case

Consider inverse transform

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

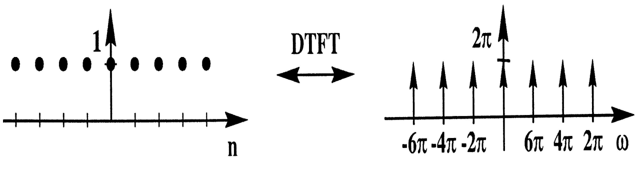
What function $X(e^{j\omega})$ would yield $x(n) \equiv 1$?

Let $X(e^{j\omega}) = 2\pi\delta(\omega),\ -\pi \leq \omega \leq \pi,$

then $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega)e^{j\omega n}d\omega = 1$ (again by sifting property).

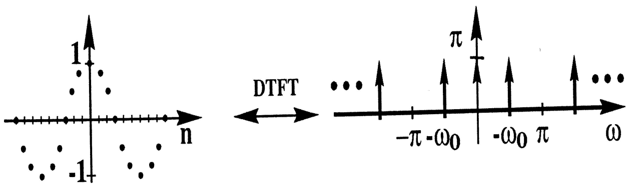
Note that $X(e^{j\omega})$ must be periodic with period 2π, so we have:

$$X(e^{j\omega}) = 2\pi\sum_k \delta(\omega - 2\pi k)$$



3. $e^{j\omega_0 n} \overset{DTFT}{\leftrightarrow} rep_{2\pi}[2\pi\delta(\omega-\omega_0)]$ (by modulation property)

4. $\cos(\omega_0 n) \overset{DTFT}{\leftrightarrow} rep_{2\pi}[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)]$



5. $x(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{else} \end{cases}$

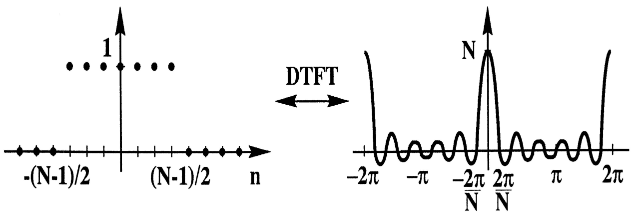
$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\ &= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \\ &= e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)} \end{aligned}$$

6. $y(n) = \begin{cases} 1, & -(N-1)/2 \leq n \leq (N-1)/2 \\ 0, & \text{else} \end{cases}$

N is odd
$y(n) = x(n + (N-1)/2)$ where $x(n)$ is signal from Example 5

$$Y(e^{j\omega}) = X(e^{j\omega})e^{j\omega(N-1)/2} \quad \text{(by sifting property)}$$
$$= \left[e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right]e^{j\omega(N-1)/2}$$

$$Y(e^{j\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$



What happens as $N \to \infty$?

$y(n) \to 1$

$-\infty < n < \infty$

$2\pi/N \to 0$

$Y(e^{j\omega}) \to rep_{2\pi}[2\pi\delta(\omega)]$

$\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})d\omega = y(0)$

DTFT and DT LTI systems

Recall that in CT case we obtained a general characterization in terms of CTFT by expressing x(t) as a superposition of complex exponential signals and then using frequency response H(f) to determine response to each such signal.

Here we take a different approach.

For any DT LTI system, we know that input and output are related by

$$y(n) = \sum_k h(n-k)x(k)$$

Thus

$$\begin{aligned}
Y(e^{j\omega}) &= \sum_n y(n)e^{-j\omega n} \\
&= \sum_n \sum_k h(n-k)x(k)e^{-j\omega n} \\
&= \sum_k \left\{ \sum_n h(n-k)e^{-j\omega n} \right\} x(k) \\
&= \sum_k H(e^{j\omega})e^{-j\omega k}x(k) \quad \text{(by sifting property)}
\end{aligned}$$

where $H(e^{j\omega})$ is the DTFT of the impulse response h(n).
Rearranging,

$$\begin{aligned}
Y(e^{j\omega}) &= H(e^{j\omega}) \sum_k x(k)e^{-j\omega k} \\
&= H(e^{j\omega})X(e^{j\omega})
\end{aligned}$$

How is $H(e^{j\omega})$ related to the frequency response?
Consider

$$x(n) = e^{j\omega_0 n}$$

$$X(e^{j\omega}) = rep_{2\pi}[2\pi\delta(\omega - \omega_0)]$$

$$X(e^{j\omega}) + 2\pi \sum_k \delta(\omega - \omega_0 - 2\pi k)$$

$$\begin{aligned}
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
&= 2\pi \sum_k H(e^{j\omega_0 = 2\pi k})\delta(\omega - \omega_0 - 2\pi k) \\
&= H(e^{j\omega_0})2\pi \sum_k \delta(\omega - \omega_0 - 2\pi k)
\end{aligned}$$

$$Y(e^{j\omega}) = H(e^{j\omega_0})X(e^{j\omega})$$

$$\therefore y(n) = H(e^{j\omega_0})x(n)$$

so $H(e^{j\omega})$ is also the frequency response of the DT system.

We thus have two equivalent characterizations for the response y(n) of a DT LTI system to any input x(n).

$$y(n) = \sum_k h(n-k)x(k)$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Convolution Theorem

Since x(n) and h(n) are arbitrary signals, we also have the following transform relation:

$$\sum_k x_1(k)x_2(n-k) \overset{DTFT}{\leftrightarrow} X_1(e^{j\omega})X_2(e^{j\omega})$$

or

$$x_1(n) * x_2(n) \overset{DTFT}{\leftrightarrow} X_1(e^{j\omega})X_2(e^{j\omega})$$

Product theorem

As mentioned earlier, we do not have a reciprocity relation for the DT case.

However, by direct evaluation of the DTFT, we also have the following result:

$$x_1(n)x_2(n) \overset{DTFT}{\leftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j(\omega-\mu)})X_2(e^{j\mu})d\mu$$

Note that this is periodic convolution.

Chapter 4

Sampling

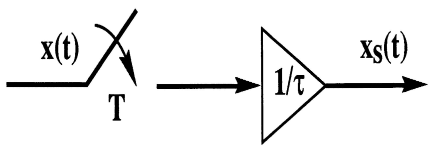
4.1 Chapter Outline

In this chapter, we will discuss:

1. Analysis of Sampling
2. Relation between CTFT and DTFT
3. Sampling Rate Conversion (Scaling in DT)

4.2 Analysis of Sampling

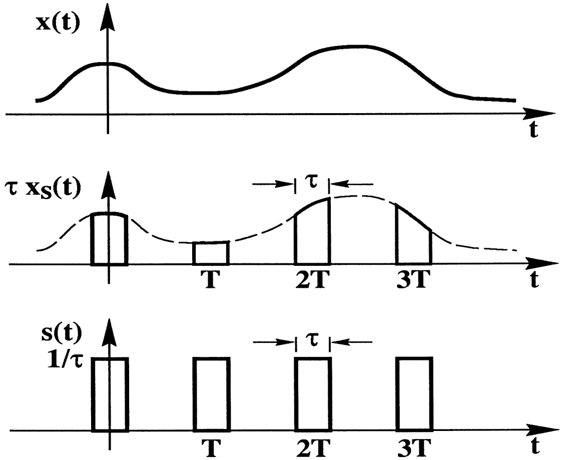
A simple scheme for sampling a waveform is to *gate* it.



T - period

τ - interval for which switch is closed

τ/T - duty cycle
$x_s(t) = s(t)x(t)$

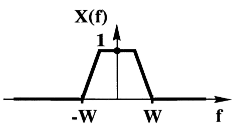


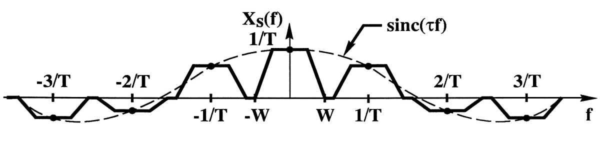
Fourier Analysis

$X_s(f) = S(f) * X(f)$
$s(t) = \text{rep}_T\left[\frac{1}{\tau}\text{rect}\left(\frac{t}{\tau}\right)\right]$

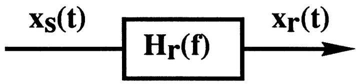
$$\begin{aligned} S(f) &= \frac{1}{T}\text{comb}_{\frac{1}{T}}[\text{sinc}(\tau f)] \\ &= \frac{1}{T}\sum_k \text{sinc}(\tau k/T)\delta(f - k/T) \end{aligned}$$

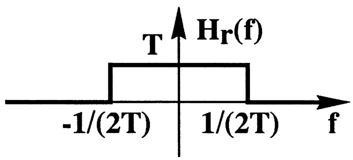
$X_s(f) = \frac{1}{T}\sum_k \text{sinc}(\tau k/T)X(f - k/T)$





How do we reconstruct x(t) from $x_s(t)$?





Nyquist condition

Perfect reconstruction of x(t) from $x_s(t)$ is possible if
$X(f) = 0, \quad |f| \geq 1/(2T)$

Nyquist sampling rate

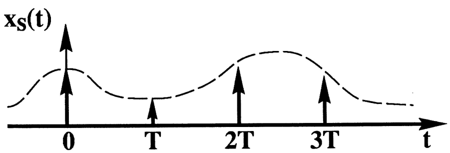
$$f_s = \frac{1}{T} = 2W$$

Ideal Sampling

What happens as $\tau \to 0$?

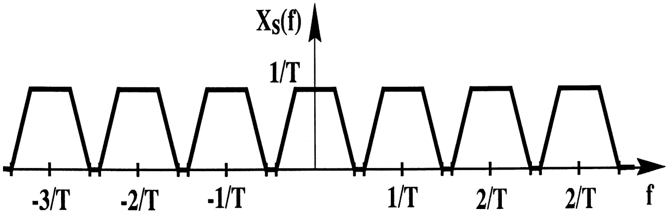
$$s(t) \to \sum_m \delta(t - mT)$$

$$x_s(t) \to \sum_m x(mT)\delta(t - mT) = \text{comb}_T[x(t)]$$



$$X_s(f) \to \frac{1}{T}\sum_k X(f - k/T) = \frac{1}{T} rep_{\frac{1}{T}}[X(f)]$$

- An ideal lowpass filter will again reconstruct x(t)
- In the sequel, we assume ideal sampling



Transform Relations

$$\text{rep}_T[x(t)] \overset{CTFT}{\leftrightarrow} \frac{1}{T}\text{comb}_{\frac{1}{T}}[X(f)]$$

$$\text{comb}_T[x(t)] \overset{CTFT}{\leftrightarrow} \frac{1}{T}\text{rep}_{\frac{1}{T}}[X(f)]$$

Given one relation, the other follows by reciprocity.

Whittaker-Kotelnikov-Shannon Sampling Expansion

$X_r(f) = H_r(f)X_s(f)$

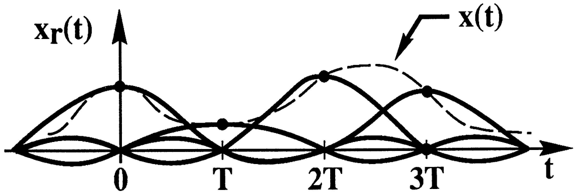
$H_r(f) = T\text{rect}(Tf)$

$x_r(t) = h_r(t) * x_s(t)$

$h_r(t) = \text{sinc}(t/T)$

$x_s(t) = \sum_m x(mT)\delta(t - mT)$

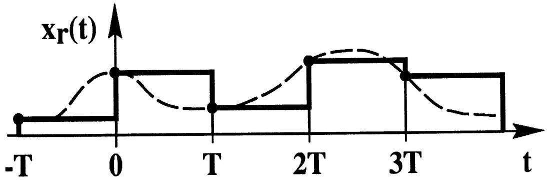
$x_r(t) = \sum_m x(mT)\text{sinc}\left(\frac{t - mT}{T}\right)$



$$\begin{aligned} x_r(nT) &= \sum_m x(mT)\text{sinc}(n - m) \\ &= x(nT) \end{aligned}$$

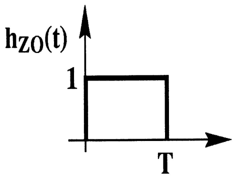
If Nyquist condition is satisfied, $x_r(t) \equiv x(t)$.

Zero Order Hold Reconstruction

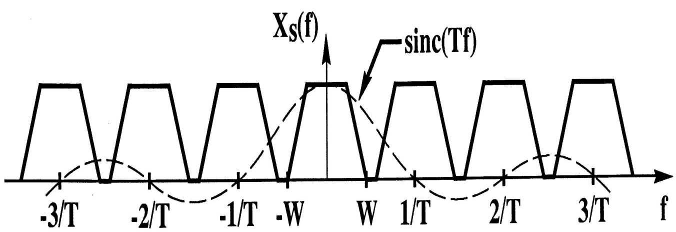


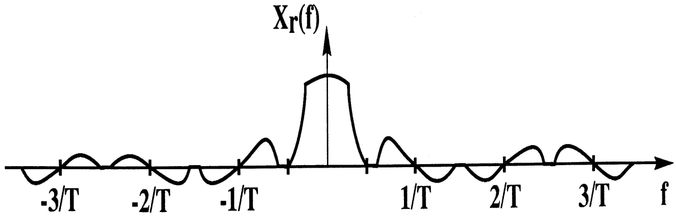
$$
\begin{aligned}
x_r(t) &= \sum_m x(mT)\mathrm{rect}\left(\frac{t - T/2 - mT}{T}\right) \\
&= h_{ZO}(t) * x_s(t)
\end{aligned}
$$

$$h_r(t) = \mathrm{rect}\left(\frac{t - T/2}{T}\right)$$

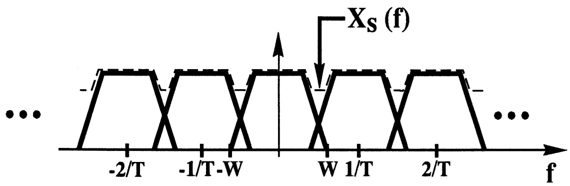


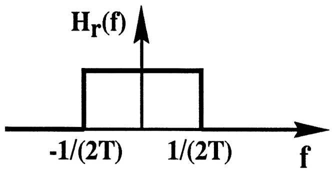
$X_r(f) = H_{ZO}(f)X_s(f)$
$H_{ZO}(f) = T\mathrm{sinc}(Tf)e^{-j2\pi f(T/2)}$

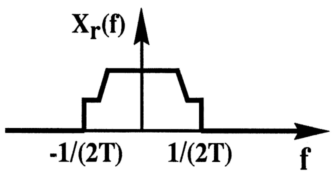


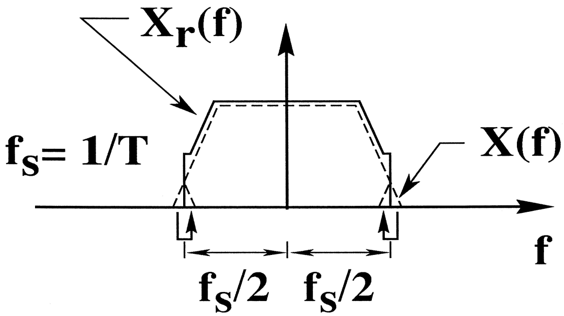


- One way to overcome the limitations of zero order hold reconstruction is to oversample, i.e. choose $f_s = 1/T >> W$
- This moves spectral replications farther out to where they are better attenuated by sinc(Tf)
- What happens if we inadvertently undersample, and then reconstruct with an ideal lowpass filter?









Effect of undersampling

- frequency truncation error

$$X_r(f) = 0, \qquad |f| \geq f_s/2$$

- aliasing error
  Frequency components at $f_1 = f_s/2 + \Delta$ fold down to and mimic frequencies $f_2 = f_s/2 - \Delta$.

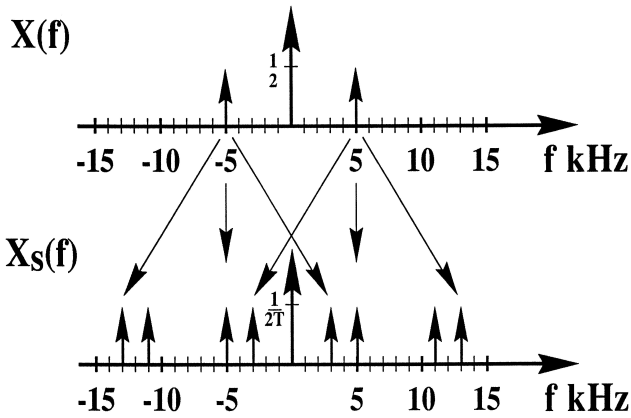
Example

- $x(t) = \cos[2\pi(5000)t]$
- sample at $f_s = 8\ kHz$
- reconstruct with ideal lowpass filter having cutoff at 4 kHz

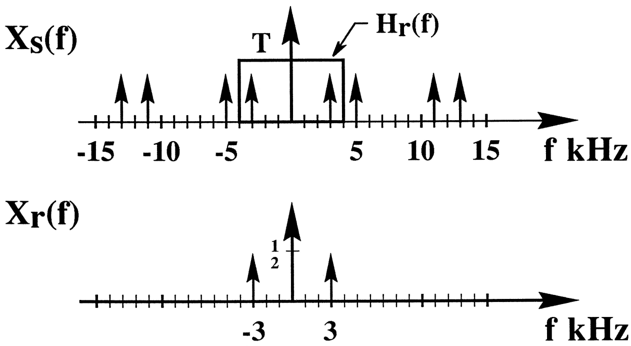
Sampling

$x_r(t) = \cos[2\pi(3000)t]$
Note that x(t) and $x_r(t)$ will have the same sample values at times $t = nT$ (T=1/(8000)).

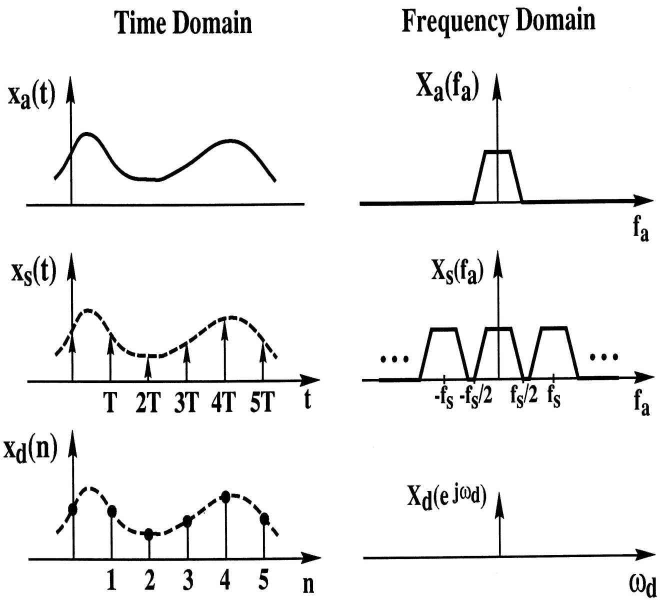
$$\begin{aligned} x(nT) &= \cos\{2\pi(5000)n/8000\} \\ &= \cos\{2\pi(5000)n/8000\} \\ &= \cos\{2\pi[(5000)n0(8000)n]/8000\} \\ &= \cos\{-2\pi(3000)n/8000\} \\ &= x_r(nT) \end{aligned}$$



Reconstruction



4.3 Relation between CTFT and DTFT



We have already shown that

$X_s(f_a) = \frac{1}{T} rep_{\frac{1}{T}}[X(f_a)]$

Suppose we evaluate the CTFT of $x_s(t)$ directly

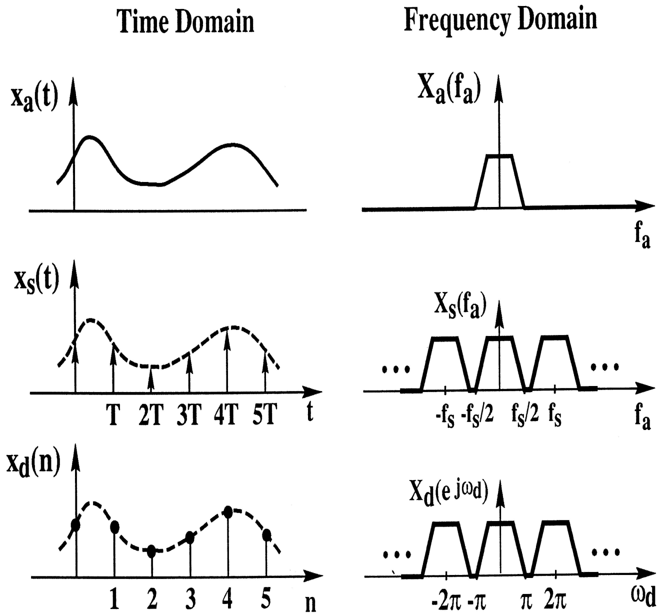
$$\begin{aligned} X_s(f_a) &= F\left\{\sum_n x_a(nT)\delta(t-nT)\right\} \\ &= \sum_n x_a(nT) F\{\delta(t-nT)\} \end{aligned}$$

$$X_s(f_a) = \sum_n x_a(nT) e^{-j2\pi f_a nT}$$

Recall

$$X_d(e^{j\omega_d}) \equiv \sum_n x_d(n) e^{-j\omega_d n}$$

But $x_d(n) = x_a(nT)$.
Let $\omega_d = 2\pi f_a T = 2\pi(f_a/f_s)$.
Rearranging as $f_a = \left(\frac{\omega_d}{2\pi}\right) f_s$, we obtain
$X_d(e^{j\omega_d}) = X_s\left[\left(\frac{\omega_d}{2\pi}\right) f_s\right]$



Example

- CT Analysis

  $x_a(t) = \cos(2\pi f_{a0} t)$

  $X_a(f_a) = \frac{1}{2}\left[\delta(f_a - f_{a0}) + \delta(f_a + f_{a0})\right]$

$$\begin{aligned} X_s(f_a) &= f_s \text{rep}_{f_s}[X_a(f_a)] \\ &= \frac{f_s}{2} \sum_k \left[\delta(f_a - f_{a0} - kf_s) + \delta(f_a + f_{a0} - kf_s)\right] \end{aligned}$$

- DT Analysis

$$\begin{aligned} x_d(n) &= x_a(nT) \\ &= \cos(2\pi f_{a0} nT) \\ &= \cos(\omega_{d0} n) \end{aligned}$$

$$\omega_{d0} = 2\pi f_{a0} T = 2\pi (f_{a0}/f_s)$$

$$X(e^{j\omega_d}) = \pi \sum_k \left[\delta(\omega_d - \omega_{d0} - 2\pi k) + \delta(\omega_d + \omega_{d0} - 2\pi k)\right]$$

- Relation between CT and DT Analyses

$$X_d(e^{j\omega_d}) = X_s\left[\left(\frac{\omega_d}{2\pi}\right) f_s\right]$$

$$X_s(f_a) = \frac{f_s}{2} \sum_k [\delta(f_a - f_{a0} - kf_s) + \delta(f_a + f_{a0} - kf_s)]$$

$$X_d(e^{j\omega_d}) = \frac{f_s}{2} \sum_k \left[\delta\left(\frac{\omega_d f_s}{2\pi} - f_{a0} - kf_s\right) + \delta\left(\frac{\omega_d f_s}{2\pi} + f_{a0} - kf_s\right)\right]$$

Recall $\delta(ax+b) = \frac{1}{|a|}\delta(x + b/a)$.

$$\begin{aligned} X_d(e^{j\omega_d}) &= \frac{f_s}{2} \sum_k \left[\frac{2\pi}{f_s}\delta\left(\omega_d - \frac{2\pi f_{a0}}{f_s} - \frac{k2\pi f_s}{f_s}\right)\right] \\ &+ \left[\frac{2\pi}{f_s}\delta\left(\omega_d + \frac{2\pi f_{a0}}{f_s} - \frac{k2\pi f_s}{f_s}\right)\right] \\ &= \pi \sum_k [\delta(\omega_d - \omega_{d0} - 2\pi k) + \delta(\omega_d + \omega_{d0} - 2\pi k)] \end{aligned}$$

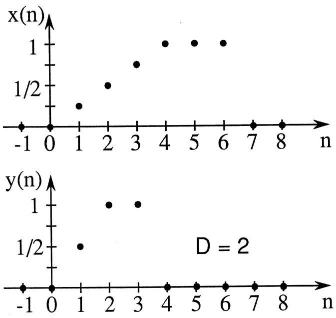
Aliasing

- CT
  $f_{a1} = f_s/2 + \Delta_a$ folds down to $f_{a2} = f_s/2 - \Delta_a$.

- DT
  Let $\omega_d = 2\pi\left(\frac{f_a}{f_s}\right) \quad \Delta_d = 2\pi\left(\frac{\Delta_a}{f_s}\right)$
  $\omega_{d1} = \pi + \omega_d$ is identical to $\omega_{d2} = \pi - \Delta_d$.

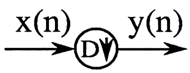
4.4 Sampling Rate Conversion

Recall definitions for DT scaling.



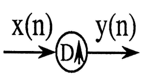
Downsampling

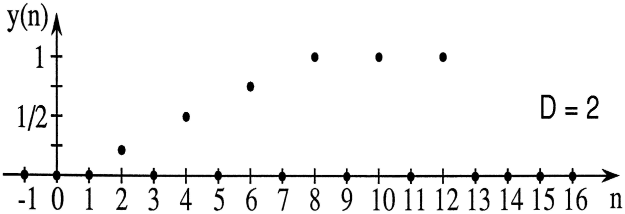
$y(n) = x(Dn)$



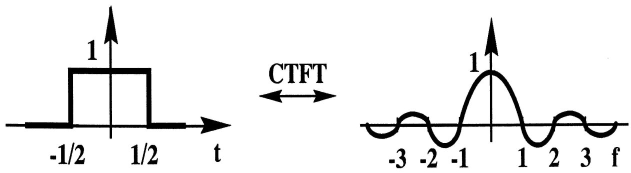
Upsampling

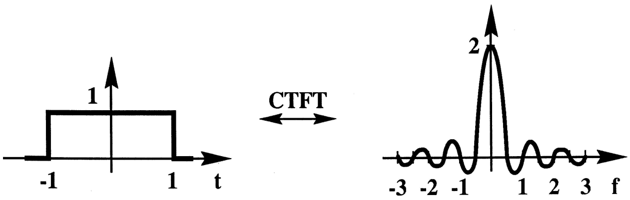
$$y(n) = \begin{cases} x(n/D), & \text{if n/D is an integer} \\ 0, & \text{else} \end{cases}$$





In CT, we have a very simple transform relation for scaling:
$x(at) \overset{CTFT}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right)$





What is the transform relation for DT?

1. As in definitions for downsampling and upsampling, we must treat these cases separately
2. Relations will combine elements from both *scaling* and *sampling* transform relations for CT

Downsampling

$y(n) = x(Dn)$

$Y(e^{j\omega}) = \sum_n x(Dn) e^{-j\omega n}$

let $m = Dn \;\Rightarrow n = m/D$

$Y(e^{j\omega}) = \sum_m x(m) e^{-j\omega m/D}$ (m/D is an integer)

To remove restrictions on m, define a sequence:

$s_D(m) = \begin{cases} 1, & \text{m/D is an integer} \\ 0, & \text{else} \end{cases}$

then

$$Y(e^{j\omega}) = \sum_m s_D(m)x(m)e^{j\omega m/D}$$

Alternate expression for $s_D(m)$:

$D = 2$

$$s_2(m) = \frac{1}{2}[1 + (-1)^m]$$
$$= \frac{1}{2}[1 + e^{-j2\pi m/2}]$$

$s_2(0) = 1$

$s_2(1) = 0$

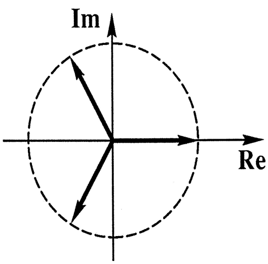
$s_2(m + 2k) = s(m)$

D=3

$s_3(m) = \frac{1}{3}[1 + e^{-j2\pi m/3} + e^{-j2\pi(2)m/3}]$

$s_3(0) = \frac{1}{3}[1 + 1 + 1] = 1$

$s_3(1) = \frac{1}{3}[1 + e^{-j2\pi/3} + e^{-j2\pi(2)/3}] = 0$



$s_3(2) = \frac{1}{3}[1 + e^{-j2\pi(2)/3} + e^{-j2\pi(4)/3}] = 0$

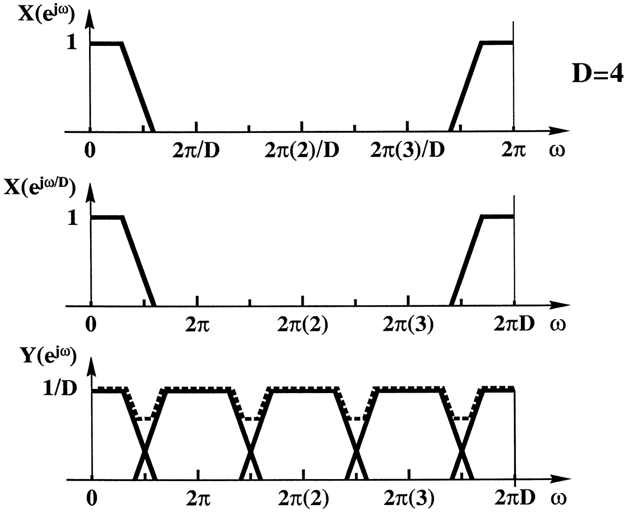
$s_3(m + 3k) = s_3(m)$

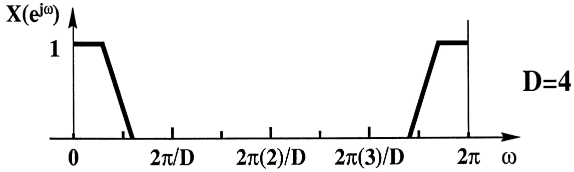
In general,

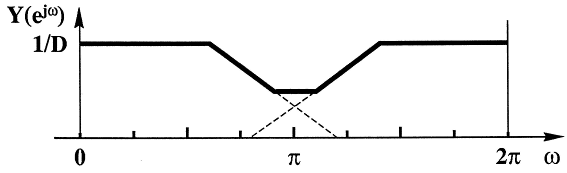
$$s_D(m) = \frac{1}{D}\sum_{k=0}^{D-1} e^{-j2\pi km/D}$$
$$= \frac{1}{D}\frac{1 - e^{-j2\pi m}}{1 - e^{-j2\pi m/D}}$$

$$s_D(m) = \begin{cases} 1, & \text{m/D is an integer} \\ 0, & \text{else} \end{cases}$$

$$\begin{aligned} Y(e^{j\omega}) &= \sum_m s_D(m) x(m) e^{-j\omega m/D} \\ &= \sum_m \frac{1}{D} \sum_{k=0}^{D-1} e^{-j2\pi k m/D} x(m) e^{-j\omega m/D} \\ &= \frac{1}{D} \sum_{k=0}^{D-1} \sum_m x(m) e^{-j[(\omega+2\pi k)/D]m} \\ &= \frac{1}{D} \sum_{k=0}^{D-1} X(e^{j(\omega+2\pi k)/D}) \end{aligned}$$



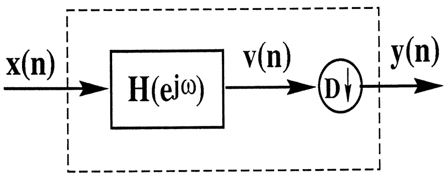


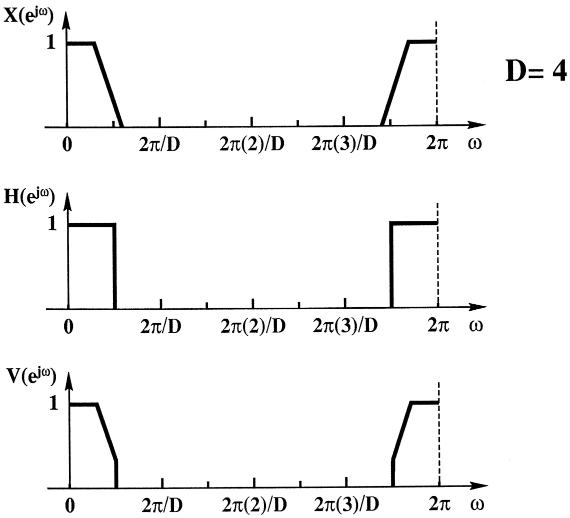


Decimator

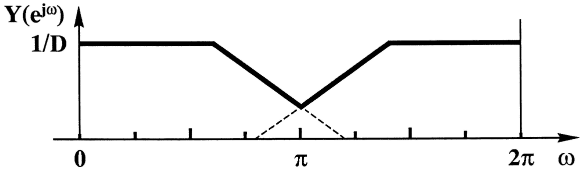
To prevent aliasing due to downsampling, we prefilter the signal to bandlimit it to highest frequency $\omega_d = \pi/D$.

The combination of a low pass filter followed by a downsampler is referred to as a *decimator.*

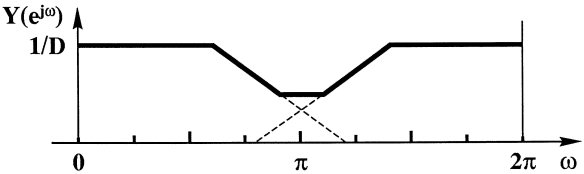




With prefilter



Without prefilter



Upsampling

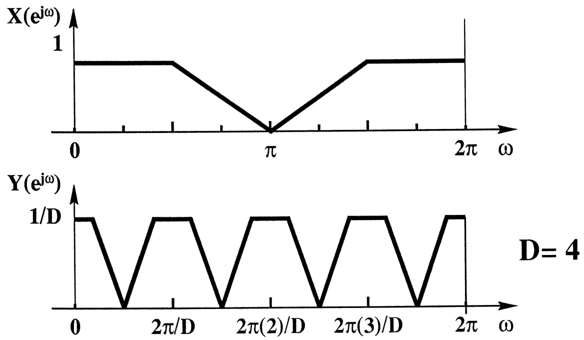
$$y(n) = \begin{cases} x(n/D), & \text{n/D is an integer} \\ 0, & \text{else} \end{cases}$$

$$\begin{aligned} Y(e^{j\omega}) &= \sum_n x(n/D)e^{-j\omega n} \quad \text{(n/D) is an integer} \\ &= \sum_n s_D(n)x(n/D)e^{-j\omega n} \end{aligned}$$

Let $m = n/D \;\Rightarrow\; n = mD$

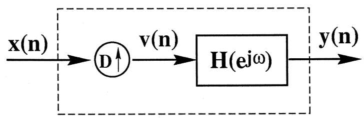
$Y(e^{j\omega}) = \sum_m s_D(mD)x(m)e^{-j\omega mD}$ but $s_D(mD) \equiv 1$

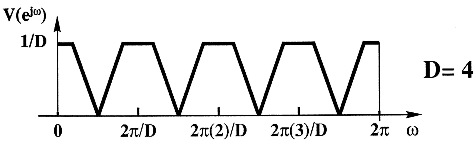
$\therefore\; Y(e^{j\omega}) = X(e^{j\omega D})$

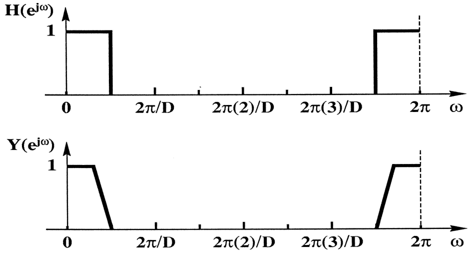


Interpolator

To interpolate between the nonzero samples generated by upsampling, we use a low pass filter with cutoff at $\omega_d = \pi/D$.
The combination of an upsampler followed by a low pass filter is referred to as an *interpolator*.







Time Domain Analysis of an Interpolator

$$y(n) = \sum_k v(k)h(n-k)$$

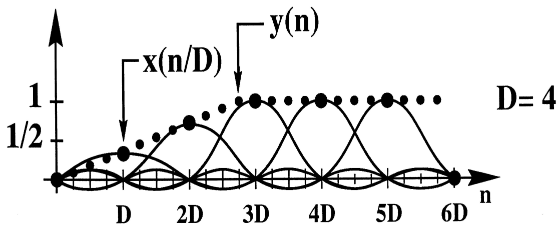
$$v(k) = s_D(k)x(k/D)$$

let $\ell = k/D \;\Rightarrow\; k = \ell D$

$$y(n) = \sum_\ell x(\ell)h(n-\ell D)$$

$$\begin{aligned} h(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/D}^{\pi/D} e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\mathrm{CTFT}^{-1}\left\{\mathrm{rect}\left(\frac{\omega}{2\pi/D}\right)\right\}\bigg|_{t=n/2\pi} \\ &= \frac{1}{2\pi}\frac{2\pi}{D}\mathrm{sinc}\left[\frac{2\pi}{D}\left(\frac{n}{2\pi}\right)\right] \\ &= \frac{1}{D}\mathrm{sinc}\left(\frac{n}{D}\right) \end{aligned}$$

$$\begin{aligned} y(n) &= \sum_\ell x(\ell)h(n-\ell D) \\ &= \sum_\ell x(\ell)\mathrm{sinc}\left(\frac{n-\ell D}{D}\right) \end{aligned}$$



Chapter 5

Z Transform (ZT)

5.1 Chapter Outline

In this chapter, we will discuss:

1. Derivation of the Z transform
2. Convergence of the ZT
3. ZT properties and pairs
4. ZT and linear, constant coefficient difference equations
5. Inverse Z transform
6. General Form of Response of LTI Systems

5.2 Derivation of the Z Transform

1. Extension of DTFT
2. Transform exists for a larger class of signals than does DTFT
3. Provides an important tool for characterizing behavior of LTI systems with rational transfer functions

- Recall sufficient conditions for existence of the DTFT

1. $\sum_n |x(n)|^2 < \infty$

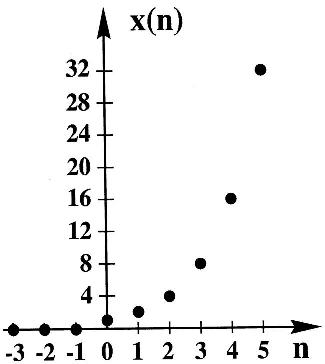
   or

2. $\sum_n |x(n)| < \infty$

- By using impulses we were able to define the DTFT for periodic signals which satisfy neither of the above conditions

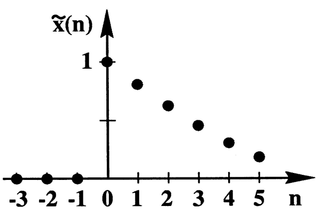
Example 1

Consider $x(n) = 2^n u(n)$



It also satisfies neither condition for existence of the DTFT.

Define $\tilde{x}(n) = r^{-n} x(n), \qquad r > 2$



Now

$$\sum_{n=0}^{\infty} |\widetilde{x}(n)| = \sum_{n=0}^{\infty} |r^{-n} x(n)|$$
$$= \sum_{n=0}^{\infty} (2/r)^n$$

$$\sum_{n=0}^{\infty} |\widetilde{x}(n)| = \frac{1}{1-2/r}, \ (2/r) = 1 \text{ or } r > 2$$

$\therefore$ DTFT of $\widetilde{x}(n)$ exists

$$\widetilde{X}(e^{j\omega}) = \sum_{n=0}^{\infty} \widetilde{x}(n) e^{-j\omega n}$$

$$\widetilde{X}(e^{j\omega}) = \sum_{n=0}^{\infty} [2(re^{j\omega})^{-1}]^n$$
$$= \frac{1}{1-2(re^{j\omega})^{-1}}, \ |2(re^{j\omega})^{-1}| < 1 \text{ or } r > 2$$

Now let's express $\widetilde{X}(e^{j\omega})$ in terms of x(n)

$$\widetilde{X}(e^{j\omega}) = \sum_{n=0}^{\infty} r^{-n} x(n) e^{-j\omega n}$$
$$= \sum_{n=0}^{\infty} x(n) (re^{j\omega})^{-n}$$

Let $z = re^{j\omega}$ and define the Z transform (ZT) of x(n) to be the DTFT of x(n) after multiplication by the convergence factor $r^{-n}u(n)$.

$$X(z) = \widetilde{X}(e^{j\omega}) = \sum_n x(n) z^{-n}$$

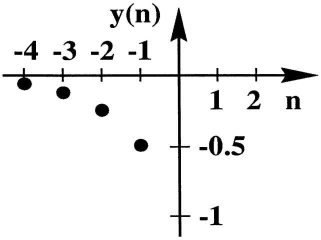
For the example $x(n) = 2^n u(n)$.

$$X(z) = \frac{1}{1-2z^{-1}}, \qquad |z| > 2$$

It is important to specify the region of convergence since the transform is not uniquely defined without it.

Example 2

Let $y(n) = -2^n u(-n-1)$



$$Y(z) = \sum_n y(n) z^{-n}$$
$$= -\sum_{n=-\infty}^{-1} 2^n z^{-n}$$

$$Y(z) = -\sum_{n=-\infty}^{-1} (z/2)^{-n}$$
$$= -\sum_{n=1}^{\infty} (z/2)^n$$
$$= -\sum_{n=0}^{\infty} (z/2)^n + 1$$

$$Y(z) = 1 - \frac{1}{1 - z/2}, \ |z/2| < 1 \ or \ |z| < 2$$

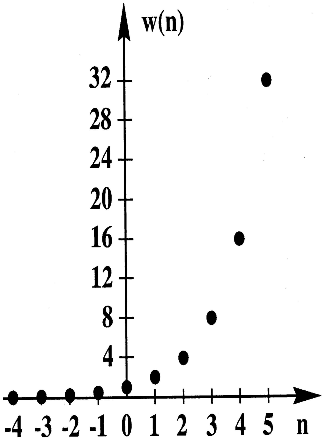
$$Y(z) = \frac{-z/2}{1 - z/2}$$
$$= \frac{1}{1 - 2z^{-1}}, \ |z| < 2$$

So we have
$x(n) = 2^n u(n) \overset{ZT}{\leftrightarrow} X(z) = \frac{1}{1 - 2z^{-1}}, \ |z| > 2$
$y(n) = -2^n u(-n-1) \overset{ZT}{\leftrightarrow} Y(z) = \frac{1}{1 - 2z^{-1}}, \ |z| < 2$

1. The two transforms have the same functional form
2. They differ only in their regions of convergence

Example 3

$w(n) = 2^n, \quad -\infty < n < \infty$



$w(n) = x(n) - y(n)$

By linearity of the ZT,

$$\begin{aligned} W(z) &= X(z) - Y(z) \\ &= \frac{1}{1 - 2z^{-1}} - \frac{1}{1 - 2z^{-1}} = 0 \end{aligned}$$

But note that $X(z)$ and $Y(z)$ have no common region of convergence. Therefore, there is no ZT for

$w(n) = 2^n, \ -\infty < n < \infty$

5.3 Convergence of the ZT

1. A series

$$\sum_{n=0}^{\infty} u_n$$

is said to converge to U if given any real $\epsilon > 0$, there exists an integer M such that
$\left| \sum_{n=0}^{N-1} u_n - U \right| < \epsilon$ for all $N > M$

2. Here the sequence u_n and the limit U may be either real or complex

3. A sufficient condition for convergence of the series $\sum_{n=0}^{\infty} u_n$ is that of *absolute convergence* i.e.

$$\sum_{n=0}^{\infty} |u_n| < \infty$$

4. Note that absolute convergence is not *necessary* for ordinary convergence as illustrated by the following example
The series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n} = -\ln(2)$$

is convergent, whereas the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not.

5. However, when we discuss convergence of the Z transform, we consider only *absolute* convergence Thus, we say that

$$X(z) = \sum_n x(n) z^{-n}$$

converges at $z = z_0$ if

$$\sum_n \left| x(n) z_0^{-n} \right| = \sum_n |x(n)| r_0^{-n} < \infty$$

where $r_0 = |z_0|$

Region of Convergence for ZT

As a consequence of restricting ourselves to absolute convergence, we have the following properties:

P1. If $X(z)$ converges at $z = z_0$, then it converges for all z for which $|z| = r_0$, where $r_0 = |z_0|$.

Proof:

$$\begin{aligned}\sum_n \left|x(n)z^{-n}\right| &= \sum_n |x(n)| r_0^{-n} \\ &< \infty \text{ by hypothesis}\end{aligned}$$

P2. If $x(n)$ is a causal sequence, i.e. $x(n) = 0,\ n < 0$, and $X(z)$ converges for $|z| = r_1$, then it converges for all Z such that $|z| = r > r_1$.

Proof:

$$\begin{aligned}\sum_{n=0}^{\infty} \left|x(n)z^{-n}\right| &= \sum_{n=0}^{\infty} |x(n)| r^{-n} \\ &< \sum_{n=0}^{\infty} |x(n)| r_1^{-n} \\ &< \infty \text{ by hypothesis}\end{aligned}$$

P3. If $x(n)$ is an anticausal sequence, i.e. $x(n) = 0,\ n > 0$ and $X(z)$ converges for $|z| = r_2$, then it converges for all z such that $|z| = r < r_2$.

Proof:

$$\begin{aligned}\sum_{n=-\infty}^{0} \left|x(n)z^{-n}\right| &= \sum_{n=0}^{\infty} |x(-n)| r^{n} \\ &< \sum_{n=0}^{\infty} |x(-n)| r_2^{n} \\ &< \infty \text{ by hypothesis}\end{aligned}$$

P4. If $x(n)$ is a mixed causal sequence, i.e. $x(n) \neq 0$ for some $n < 0$ and $x(n) \neq 0$ for some $n > 0$, and $X(z)$ converges for some $|z| = r_0$, then there exists two positive reals r_1 and r_2 with $r_1 < r_0 < r_2$ such that $X(z)$ converges for all z satisfying $r_1 < |z| < r_2$.

Proof:

Let $x(n) = x_-(n) + x_+(n)$ where $x_-(n)$ is anticausal and $x_+(n)$ is causal. Since $X(z)$ converges for $|z| = r_0$, $X_-(z)$ and $X_+(z)$ must also both converge for $|z| = r_0$.

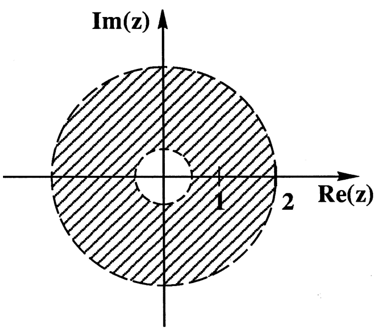
From properties 2 and 3, there exist two positive reals r_1 and r_2 with $r_1 < r_0 < r_2$ such that $X_-(z)$ converges for $|z| < r_2$ and $X_+(z)$ converges for $|z| > r_1$.
The ROC for $X(z)$ is just the intersection of these two ROC's.

Example 3

$x(n) = 2^{-|n|}u(n)$

$$\begin{aligned}
X(z) &= \sum_n 2^{-|n|} z^{-n} \\
&= \sum_{n=-\infty}^{-1} 2^n z^{-n} + \sum_{n=0}^{\infty} 2^{-n} z^{-n} \\
&= \sum_{n=0}^{\infty} (2^{-1}z)^n - 1 + \sum_{n=0}^{\infty} (2^{-1}z^{-1})^n \\
&= \frac{1}{1 - 2^{-1}z} - 1 + \frac{1}{1 - 2^{-1}z^{-1}} \\
& \quad |2^{-1}z| < \infty \qquad |2^{-1}z^{-1}| < \infty \\
& \quad |z| < 2 \qquad\qquad |z| > 1/2
\end{aligned}$$

Combining everything
$X(z) = \dfrac{-3z^{-1}/2}{1 - 5z^{-1}/2 + z^{-2}}, \qquad 1/2 < |z| < 2$



5.4 ZT Properties and Pairs

Transform Relations

1. Linearity

$$a_1 x_1(n) + a_2 x_2(n) \overset{ZT}{\leftrightarrow} a_1 X_1(z) + a_2 X_2(z)$$

2. Shifting

$$x(n - n_0) \overset{ZT}{\leftrightarrow} z^{-n_0} X(z)$$

3. Modulation

$$z_0^n x(n) \overset{ZT}{\leftrightarrow} X(z/z_0)$$

4. Multiplication by time index

$$nx(n) \overset{ZT}{\leftrightarrow} -z\frac{d}{dz}[X(z)]$$

5. Convolution

$$x(n) * y(n) \overset{ZT}{\leftrightarrow} X(z)Y(z)$$

6. Relation to DTFT
   If $X_{ZT}(z)$ converges for $|z| = 1$, then the DTFT of x(n) exists and

$$X_{DTFT}(e^{j\omega}) = X_{ZT}(z)\big|_{z=e^{j\omega}}$$

Important Transform Pairs

1. $\delta(n) \overset{ZT}{\leftrightarrow} 1, \qquad \text{all z}$

2. $a^n u(n) \overset{ZT}{\leftrightarrow} \dfrac{1}{1 - az^{-1}}, \qquad |z| > a$

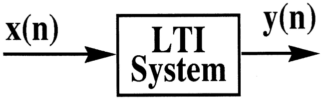
3. $-a^n u(-n-1) \overset{ZT}{\leftrightarrow} \dfrac{1}{1 - az^{-1}}, \qquad |z| < a$

4. $na^n u(n) \overset{ZT}{\leftrightarrow} \dfrac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| > a$

5. $-na^n u(-n-1) \overset{ZT}{\leftrightarrow} \dfrac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| < a$

5.5 ZT,Linear Constant Coefficient Difference Equations

1. From the convolution property, we obtain a characterization for all LTI systems



   - impulse response: $y(n) = h(n) * x(n)$
   - transfer function: $Y(z) = H(z)X(z)$

2. An important class of LTI systems are those characterized by linear, constant coefficient difference equations

$$y(n) = \sum_{k=0}^{M} a_k x(n-k) - \sum_{\ell=1}^{N} h_\ell y(n-\ell)$$

   - nonrecursive
     $N = 0$
     always finite impulse response (FIR)
   - recursive
     $N > 0$
     usually infinite impulse response (IIR)

3. Take ZT of both sides of the equation

$$y(n) = \sum_{k=0}^{M} a_k x(n-k) - \sum_{\ell=1}^{N} b_\ell y(n-\ell)$$

$$Y(z) = \sum_{k=0}^{M} a_k z^{-k} X(z) - \sum_{\ell=1}^{N} b_\ell z^{-\ell} Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\displaystyle\sum_{k=0}^{M} a_k z^{-k}}{1 + \displaystyle\sum_{\ell=1}^{N} b_\ell z^{-\ell}}$$

4. $M \geq N$
   Multiply numerator and denominator by z^M.

$$H(z) = \frac{\sum_{k=0}^{M} a_k z^{M-k}}{z^{M-N}\left[z^N + \sum_{\ell=1}^{N} b_\ell z^{N-\ell}\right]}$$

5. $M < N$
   Multiply numerator and denominator by z^N.

$$H(z) = \frac{z^{N-M}\sum_{k=0}^{M} a_k z^{M-k}}{z^N + \sum_{\ell=1}^{N} b_\ell z^{N-\ell}}$$

6. By the fundamental theorem of algebra, the numerator and denominator polynomials may always be factored

   - $M \geq N$

     $$H(z) = \frac{\prod_{k=1}^{M}(z - z_K)}{z^{M-N}\prod_{\ell=1}^{N}(z - p_\ell)}$$

   - $M < N$

     $$H(z) = \frac{z^{N-M}\prod_{k=1}^{M}(z - z_k)}{\prod_{\ell=1}^{N}(z - p_\ell)}$$

7. Roots of the numerator and denominator polynomials:

   - zeros $z_1, ..., z_M$
   - poles $p_1, ..., p_N$
   - If $M \geq N$, have $M - N$ additional poles at $|z| = \infty$
   - If $M < N$, have $N - M$ additional zeros at $z = 0$

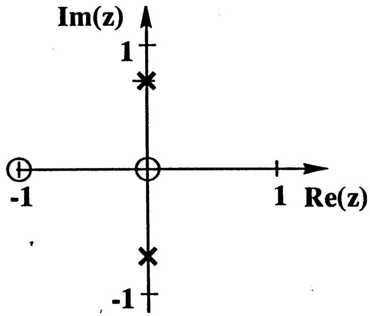
8. The poles and zeros play an important role in determining system behavior

Example

$$y(n) = x(n) + x(n-1) - \frac{1}{2}y(n-2)$$

$$Y(z) = X(z) + z^{-1}X(z) - \frac{1}{2}z^{-2}Y(z)$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1+\frac{1}{2}z^{-2}}$$
zeros : $z_1 = 0 \quad z_2 = -1$

poles : $p_1 = j/\sqrt{2} \quad p_2 = -j/\sqrt{2}$



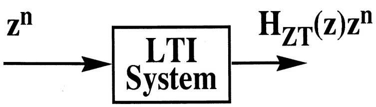
$$H(z) = \frac{z(z+1)}{z^2+1/2} = \frac{z(z+1)}{(z-j/\sqrt{2})(z+j/\sqrt{2})}$$

Effect of Poles and Zeros on Frequency Response

Frequency response $H(e^{j\omega})$

$$h(n) \overset{CTFT}{\leftrightarrow} H_{DTFT}(e^{j\omega}) = H(e^{j\omega})$$

$$h(n) \overset{ZT}{\leftrightarrow} H_{ZT}(z)$$
$$H_{DTFT}(e^{j\omega}) = H_{ZT}(e^{j\omega})$$



Let $z = e^{j\omega}$.

$\Rightarrow H(e^{j\omega}) = H_{ZT}(e^{j\omega})$
Assume $M < N$.

$$H(z) = \frac{z^{N-M}\prod_{k=1}^{M}(z - z_k)}{\prod_{\ell=1}^{N}(z - p_\ell)}$$

$$H(e^{j\omega}) = \frac{e^{j\omega(N-M)}\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{\ell=1}^{N}(e^{j\omega} - p_\ell)}$$

$$|H(e^{j\omega})| = \frac{\prod_{k=1}^{M}|e^{j\omega} - z_k|}{\prod_{\ell=1}^{N}|e^{j\omega} - p_\ell|}$$

$$\angle H(e^{j\omega}) = \omega(N-M) + \sum_{k=1}^{M}\angle(e^{j\omega} - z_k) - \sum_{\ell=1}^{N}\angle(e^{j\omega} - p_\ell)$$

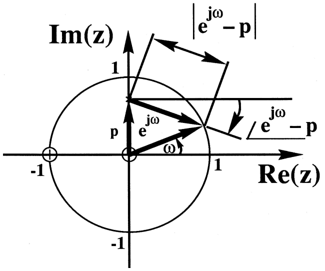
Example

$$H(z) = \frac{z(z+1)}{(z - j/\sqrt{2})(z + j/\sqrt{2})}$$

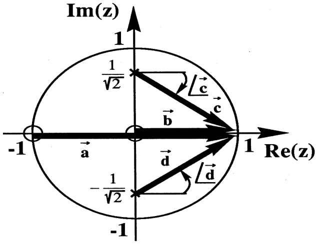
$$|H(e^{j\omega})| = \frac{|e^{j\omega} + 1|}{|e^{j\omega} - j/\sqrt{2}||e^{j\omega} + j/\sqrt{2}|}$$

$$\angle H(e^{j\omega}) = \omega + \angle(e^{j\omega} + 1) - \angle(e^{j\omega} - j/\sqrt{2}) - \angle(e^{j\omega} + j/\sqrt{2})$$

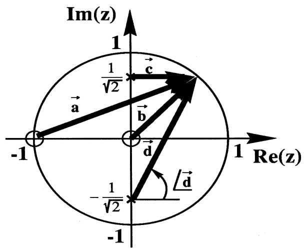
Contribution from a single pole



$\underline{\omega = 0}$

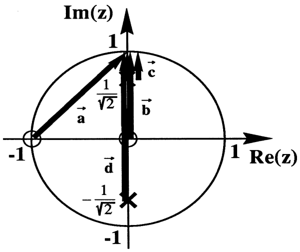


$|H(e^{j0})| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$
$|\vec{a}| = 2$
$|\vec{b}| = 1$
$|\vec{c}| = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$
$|\vec{d}| = \sqrt{\frac{3}{2}}$
$|H(e^{j0})| = \frac{4}{3} = 1.33$
$H(e^{j0}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$
$\angle\vec{a} = 0$
$\angle\vec{b} = 0$
$\angle\vec{c} = \arctan\left(\frac{1}{\sqrt{2}}\right)$
$\angle\vec{d} = -\angle\vec{c}$
$\angle H(e^{j0}) = 0$
$\underline{\omega = \pi/4}$



$|H(e^{j\pi/4})| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$
$|\vec{a}| = \sqrt{\left(1 + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} = 1.85$
$|\vec{b}| = 1$

$|\vec{c}| = \frac{1}{\sqrt{2}}$
$|\vec{d}| = \sqrt{\left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{5}{2}}$
$|H(e^{j\pi/4})| = 1.65$
$H(e^{j\pi/4}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$
$\angle\vec{a} = 22.5°$
$\angle\vec{b} = 45°$
$\angle\vec{c} = 0$
$\angle\vec{d} = 63.4°$
$\angle H(e^{j\pi/4}) = 4.1°$
$\underline{\omega = \pi/2}$



$|H(e^{j\pi/2})| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|}$
$|\vec{a}| = \sqrt{2}$
$|\vec{b}| = 1$
$|\vec{c}| = 1 - \frac{1}{\sqrt{2}}$
$|\vec{d}| = 1 + \frac{1}{\sqrt{2}}$
$|H(e^{j\pi/2})| = 2.83$
$H(e^{j\pi/4}) = \angle\vec{a} + \angle\vec{b} - \angle\vec{c} - \angle\vec{d}$
$\angle\vec{a} = 45°$
$\angle\vec{b} = 90°$
$\angle\vec{c} = 90°$
$\angle\vec{d} = 90°$
$\angle H(e^{j\pi/2}) = -45°$

General Rules

1. A pole near the unit circle will cause the frequency response to increase in the neighborhood of that pole

2. A zero near the unit circle will cause the frequency response to decrease in the neighborhood of that zero

5.6 Inverse Z Transform

1. Formally, the inverse ZT may be written as
   $x(n) = \frac{1}{j2\pi} \oint_c X(z) z^{n-1} dz$
   where the integration is performed in a counter-clockwise direction around a closed contour in the region of convergence of $X(z)$ and encircling the origin.

2. If $X(z)$ is a rational function of z, i.e. a ratio of polynomials, it is not necessary to evaluate the integral

3. Instead, we use a partial fraction expansion to express $X(z)$ as a sum of simple terms for which the inverse transform may be recognized by inspection

4. The ROC plays a critical role in this process

5. We will illustrate the method via a series of examples

Example 1

The signal $x(n) = (1/3)^n u(n)$ is input to a DT LTI system described by

$$y(n) = x(n) - x(n-1) + (1/2)y(n-1)$$

Find the output $y(n)$.

What are our options?

- direct substitution
  Assume $y(-1) = 0$.
  $y(0) = x(0) - x(-1) + (1/2)y(-1) = 1 - 0 + (1/2)(0) = 1.000$
  $y(1) = x(1) - x(0) + (1/2)y(0) = 1/3 - 1 + (1/2)(1.0) = -0.167$
  $y(2) = x(2) - x(1) + (1/2)y(1) = 1/9 - 1/3 + (1/2)(-0.167) = -0.306$

  $y(3) = x(3) - x(2) + (1/2)y(2) = 1/27 - 1/9 + (1/2)(-0.306) = -0.227$

- convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

Find impulse response by direct substitution.
$h(n) = \delta(n) - \delta(n-1) + (1/2)h(n-1)$
$h(0) = 1 - 0 + (1/2)(0) = 1$
$h(1) = 0 - 1 + (1/2)(1) = -1/2$
$h(2) = 0 - 0 + (1/2)(-1/2) = -1/4$
$h(3) = 0 - 0 + (1/2)(-1/4) = -1/8$
Recognize $h(n) = \delta(n) - (1/2)^n u(n-1)$.

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} \left[\delta(n-k) - (1/2)^{(n-k)}u(n-k-1)\right](1/3)^k u(k) \\ &= (1/3)^n u(n) - (1/2)^n \sum_{k=0}^{n-1}(2/3)^k u(n-1) \\ &= (1/3)^n u(n) - (1/2)^n \frac{1-(2/3)^n}{1-2/3} u(n) \\ &= 4(1/3)^n u(n) - 3(1/2)^n u(n) \end{aligned}$$

- DTFT

$$x(n) = (1/3)^n u(n)$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}}$$

$$h(n) = \delta(n) - (1/2)^n u(n-1)$$

$$\begin{aligned} H(e^{j\omega}) &= 1 - \left[\frac{1}{1-\frac{1}{2}e^{-j\omega}} - 1\right] \\ &= \frac{1 - e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}} \end{aligned}$$

$$\begin{aligned}
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
&= \frac{1 - e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{3}e^{-j\omega})} \\
&= \frac{1 - e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j\omega 2}}
\end{aligned}$$

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j\omega 2}} e^{j\omega n} d\omega$$

- ZT

$$x(n) = (1/3)^n u(n)$$

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$y(n) = x(n) - x(n-1) + (1/2)y(n-1)$$

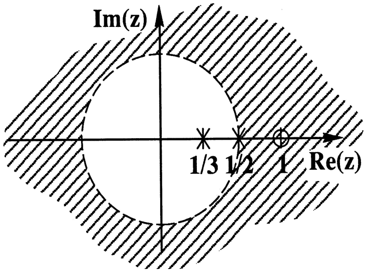
$$Y(z) = X(z) - z^{-1}X(z) + (1/2)z^{-1}Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

What is the region of convergence?
Since system is causal, $h(n)$ is a causal signal.
$\Rightarrow$ $H(z)$ converges for $|z| > 1/2$.

$$\begin{aligned}
Y(z) &= H(z)X(z) \\
&= \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}
\end{aligned}$$

$$\begin{aligned}
ROC[Y(z)] &= ROC[H(z)] \cap ROC[X(z)] \\
&= z : |z| > 1/2
\end{aligned}$$



Partial Fraction Expansion (PFE)

(Two distinct poles)

$$\frac{1-z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{3}z^{-1}}$$

To solve for A_1 and A_2, we multiply both sides by $\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)$ to obtain

$1-z^{-1} = A_1\left(1-\frac{1}{3}z^{-1}\right) + A_2\left(1-\frac{1}{2}z^{-1}\right)$

$1 = A_1 + A_2 \quad A_1 = -3$

$-1 = -\frac{1}{3}A_1 - \frac{1}{2}A_2 \quad A_2 = 4$

so

$$\frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{-3}{1-\frac{1}{2}z^{-1}} + \frac{4}{1-\frac{1}{3}z^{-1}} \text{ check}$$

$1-z^{-1} = -3\left(1-\frac{1}{3}z^{-1}\right) + 4\left(1-\frac{1}{2}z^{-1}\right)$

Now

$$y(n) = y_1(n) + y_2(n)$$

Possible ROC's

where

$Y_1(z) = \dfrac{-3}{1-\frac{1}{2}z^{-1}}, \ |z| < \dfrac{1}{2}$ or $|z| > \frac{1}{2}$

$Y_2(z) = \dfrac{4}{1-\frac{1}{3}z^{-1}}, \ |z| < \dfrac{1}{3}$ or $|z| > \frac{1}{3}$

and

$$\begin{aligned} ROC[Y(z)] &= ROC[Y_1(z)] \cap ROC[Y_2(z)] \\ &= \left\{ z : |z| > \frac{1}{2} \right\} \end{aligned}$$

Recall transform pairs:

$$a^n u(n) \overset{ZT}{\leftrightarrow} \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

$$-a^n u(-n-1) \overset{ZT}{\leftrightarrow} \frac{1}{1 - az^{-1}} \quad |z| < |a|$$

$\frac{-3}{1-\frac{1}{2}z^{-1}} \overset{ZT^{-1}}{\rightarrow} -3\left(\frac{1}{2}\right)^n u(n)$

$\frac{4}{1-\frac{1}{3}z^{-1}} \overset{ZT^{-1}}{\rightarrow} 4\left(\frac{1}{3}\right)^n u(n)$

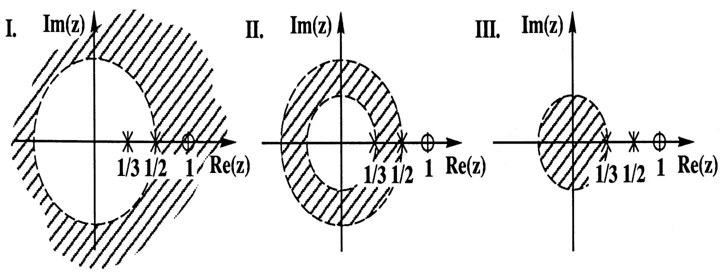
and

$$y(n) = -3\left(\frac{1}{2}\right)^n u(n) + 4\left(\frac{1}{3}\right)^n u(n)$$

Consider again

$$\begin{aligned} Y(z) &= \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} \\ &= \frac{-3}{1 - \frac{1}{2}z^{-1}} + \frac{4}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

There are 3 possible ROC's for a signal with this ZT.



Case II:

$\frac{1}{3} < |z| < \frac{1}{2}$

Possible ROC's

$Y_1(z) = \frac{-3}{1 - \frac{1}{2}z^{-1}}, \ |z| < \frac{1}{2}$ or $|z| > \frac{1}{2}$

$Y_2(z) = \frac{4}{1 - \frac{1}{3}z^{-1}}, \ |z| < \frac{1}{3}$ or $|z| > \frac{1}{3}$

and

$ROC[Y(z)] = ROC[Y_1(z)] \cap ROC[Y_2(z)]$

$\therefore \ ROC[Y_1(z)] = \left\{z : |z| < \frac{1}{2}\right\}$

$ROC[Y_2(z)] = \left\{z : |z| > \frac{1}{3}\right\}$

$$\frac{-3}{1 - \frac{1}{2}z^{-1}} \stackrel{ZT^{-1}}{\longrightarrow} -3\left(\frac{1}{2}\right)^n u(-n-1)$$

$$\frac{4}{1 - \frac{1}{3}z^{-1}} \stackrel{ZT^{-1}}{\longrightarrow} 4\left(\frac{1}{3}\right)^n u(n)$$

$$y(n) = 3\left(\frac{1}{2}\right)^n u(-n-1) + 4\left(\frac{1}{3}\right)^n u(n)$$

Case III:

$|z| < \frac{1}{3}$

$$ROC[Y_1(z)] = \left\{z : |z| < \frac{1}{2}\right\}$$

$$ROC[Y_2(z)] = \left\{z : |z| < \frac{1}{3}\right\}$$

$$\frac{-3}{1 - \frac{1}{2}z^{-1}} \stackrel{ZT^{-1}}{\longrightarrow} 3\left(\frac{1}{2}\right)^n u(-n-1)$$

$$\frac{4}{1 - \frac{1}{3}z^{-1}} \stackrel{ZT^{-1}}{\longrightarrow} -4\left(\frac{1}{3}\right)^n u(-n-1)$$

$$y(n) = 3\left(\frac{1}{2}\right)^n u(-n-1) - 4\left(\frac{1}{3}\right)^n u(-n-1)$$

Summary of Possible ROC's and Signals

$$Y(z) = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

ROC	Signal
$\frac{1}{2} < \lvert z \rvert$	$-3\left(\frac{1}{2}\right)^n u(n) + 4\left(\frac{1}{3}\right)^n u(n)$
$\frac{1}{3} < \lvert z \rvert < \frac{1}{2}$	$3\left(\frac{1}{2}\right)^n u(-n-1) + 4\left(\frac{1}{3}\right)^n u(n)$
$\lvert z \rvert < \frac{1}{3}$	$3\left(\frac{1}{2}\right)^n u(-n-1) - 4\left(\frac{1}{3}\right)^n u(-n-1)$

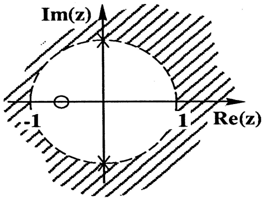
Residue Method for Evaluating Coefficients of PFE

$$Y(z) = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - \frac{1}{3}z^{-1}}$$

$$\begin{aligned}
A_1 &= \left(1 - \frac{1}{2}z^{-1}\right) Y(z)\Big|_{z=\frac{1}{2}} \\
&= \frac{1 - z^{-1}}{1 - \frac{1}{3}z^{-1}}\Big|_{z=\frac{1}{2}} \\
&= \frac{1-2}{1-2/3} \\
&= -3
\end{aligned}$$

$$\begin{aligned}
A_2 &= \left(1 - \frac{1}{3}z^{-1}\right) Y(z)\Big|_{z=\frac{1}{3}} \\
&= \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}\Big|_{z=\frac{1}{3}} \\
&= \frac{1-3}{1-3/2} \\
&= 4
\end{aligned}$$

Example 2 (complex conjugate poles)



$$Y(z) = \frac{1 + \frac{1}{2}z^{-1}}{(1 - jz^{-1})(1 + jz^{-1})}$$

$$X(z) = \frac{A_1}{1 - jz^{-1}} + \frac{A_2}{1 + jz^{-1}}$$

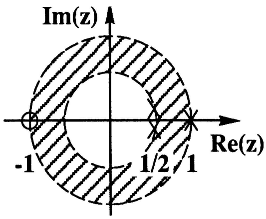
$$A_1 = \left.\frac{1 + \frac{1}{2}z^{-1}}{1 + jz^{-1}}\right|_{z=j} = \frac{1}{2}\left(1 - \frac{1}{2}j\right)$$

$$A_2 = \left.\frac{1 + \frac{1}{2}z^{-1}}{1 - jz^{-1}}\right|_{z=-j} = \frac{1}{2}\left(1 + \frac{1}{2}j\right)$$

check

$$\frac{\frac{1}{2}\left(1 - \frac{1}{2}j\right)}{1 - jz^{-1}} + \frac{\frac{1}{2}\left(1 + \frac{j}{2}\right)}{1 + jz^{-1}} = \frac{\frac{1}{2}\left(1 - \frac{1}{2}j\right)\left(1 + jz^{-1}\right) + \frac{1}{2}\left(1 + \frac{j}{2}\right)\left(1 - jz^{-1}\right)}{\left(1 - jz^{-1}\right)\left(1 + jz^{-1}\right)} = \frac{1 + \frac{1}{2}z^{-1}}{(1 - jz^{-1})(1 + jz^{-1})}$$

- Note that $A_1 = A_2^*$
- This is necessary for $y(n)$ to be real-valued
- Use it to save computation



Example 3 (poles with multiplicity greater than 1)

$$Y(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2\left(1-z^{-1}\right)}$$
$$\frac{1}{2} < |z| < 1$$

recall

$$na^n u(n) \overset{ZT}{\leftrightarrow} \frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z| > |a|$$

Try

$$\frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2\left(1-z^{-1}\right)} = \frac{A_1}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{A_2}{1-z^{-1}}$$

$$A_1 = \left.\frac{1+z^{-1}}{1-z^{-1}}\right|_{z=\frac{1}{2}} = -3$$

$$A_2 = \left.\frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}\right|_{z=1} = 8$$

check

$$\begin{aligned}\frac{-3}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{8}{1-z^{-1}} &= \frac{-3(1-z^{-1}) + 8(1-z^{-1}+\frac{1}{4}z^{-2})}{\left(1-\frac{1}{2}z^{-1}\right)^2\left(1-z^{-1}\right)} \\ &= \frac{5-5z^{-1}+2z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \\ &\neq \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2\left(1-z^{-1}\right)}\end{aligned}$$

General form of terms in PFE for pole p_ℓ with multiplicity m_ℓ

$$\frac{A_\ell, m_\ell}{(1-p_\ell z^{-1})^{m_\ell}} + \frac{A_\ell, m_\ell - 1}{(1-p_\ell z^{-1})^{m_\ell - 1}} + ... + \frac{A_\ell, 1}{(1-p_\ell z^{-1})}$$

Now we have

$$\frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2(1-z^{-1})} = \frac{A_{1,2}}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{A_{1,1}}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{A_2}{1-z^{-1}}$$

$A_{1,2} = -3, \quad A_2 = 8$

To solve for $A_{1,1}$, multiply both sides by $\left(1-\frac{1}{2}z^{-1}\right)^2$.

$$\frac{1+z^{-1}}{1-z^{-1}} = A_{1,2} + A_{1,1}\left(1-\frac{1}{2}z^{-1}\right) + A_2\frac{\left(1-\frac{1}{2}z^{-1}\right)^2}{1-z^{-1}}$$

differentiate with respect to z^{-1}

$$\frac{2}{(1-z^{-1})^2} = 0 + A_{1,1}\left(\frac{-1}{2}\right) + A_2\left(1-\frac{1}{2}z^{-1}\right)P(z)$$

let $z = \frac{1}{2}$

$$A_{1,1} = -2\frac{2}{(1-2)^2} = -4$$

Example 4

(numerator degree $\geq$ denominator degree)

$$Y(z) = \frac{1+z^{-2}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}$$

reduce degree of numerator by long division so

$$\frac{1+z^{-2}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}} = -2 + \frac{3-z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}$$

In general, we will have a polynomial of degree $M-N$.

$$\sum_{k=0}^{M-N} B_k z^{-k} \overset{ZT^{-1}}{\longrightarrow} \sum_{k=0}^{M-N} B_k \delta(n-k)$$

5.7 General Form of Response of LTI Systems

If both X(z) and H(z) are rational

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \left[\frac{P_H(z)}{Q_H(z)}\right]\left[\frac{P_X(z)}{Q_X(z)}\right] \\ &= \left[\frac{P_H(z)}{\prod_{\ell=1}^{N_H}\left(1-p_\ell^H z^{-1}\right)}\right]\left[\frac{P_X(z)}{\prod_{\ell=1}^{N_X}\left(1-p_\ell^X z^{-1}\right)}\right] \end{aligned}$$

$$Y(z) = \frac{P_Y(z)}{\prod_{\ell=1}^{N_y}\left(1-p_\ell^Y z^{-1}\right)}$$

$P_Y(z) = P_H(z)P_X(z)$

$N_Y = N_H + N_X$

p_ℓ^Y is a combined set of poles p_ℓ^H and p_ℓ^X.

Drop superscript/subscript Y.

Accounting for poles with multiplicity > 1

$$Y(z) = \frac{P(z)}{\prod_{\ell=1}^{D}\left(1-p_\ell z^{-1}\right)^{m_\ell}}$$

D - number of distinct poles

$$N = \sum_{\ell=1}^{D} m_\ell$$

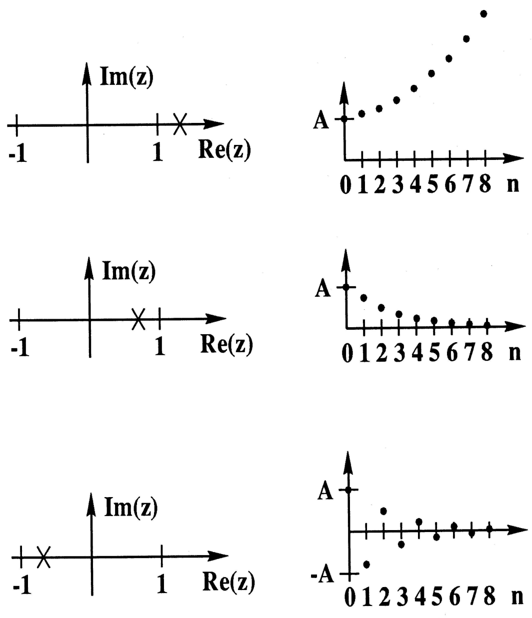
For $M < N$

$$Y(z) = \sum_{\ell=1}^{D}\sum_{k=1}^{m_\ell}\frac{A_{\ell k}}{\left(1-p_\ell z^{-1}\right)^k}$$

- Each term under summation will give rise to a term in the output y(n)
- It will be causal or anticausal depending on location of pole relative to ROC
- Poles between origin and ROC result in causal terms
- Poles separated from origin by ROC result in anticausal terms
- For simplicity, consider only causal terms in what follows

Real pole with multiplicity 1

$$\frac{A}{1 - pz^{-1}} \stackrel{ZT^{-1}}{\rightarrow} Ap^n u(n)$$



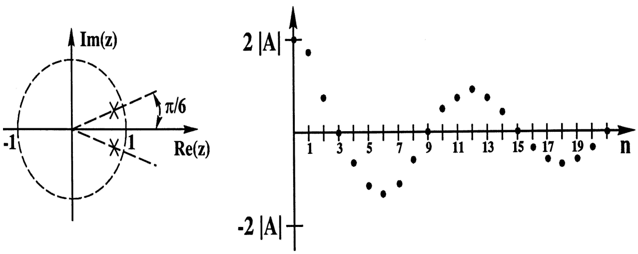
Complex conjugate pair of poles with multiplicity 1

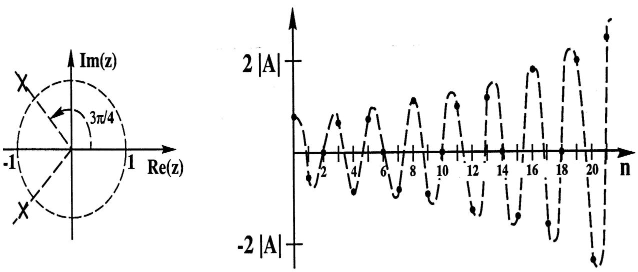
$$\frac{A}{1-pz^{-1}} + \frac{A^*}{1-p^*z^{-1}} \overset{ZT^{-1}}{\rightarrow} Ap^n u(n) + A^*(p^*)^n u(n)$$

$$\begin{aligned}[Ap^n + A^*(p^*)^n]\, u[n] &= \left[|A|e^{j\angle A}(|p|e^{j\angle p})^n + |A|e^{-j\angle A}(|p|e^{-j\angle p})^n\right] u(n) \\ &= 2|A||p|^n \cos(\angle pn + \angle A) u(n)\end{aligned}$$

- sinusoid with amplitude $2|A||p|^n$
  1. grows exponentially if $|p| > 1$
  2. constant if $p = 1$
  3. decays exponentially if $|p| < 1$
- digital frequency $\omega_d = \angle p$ radians/sample
- phase $\angle A$

Examples





Real Pole with multiplicity 2

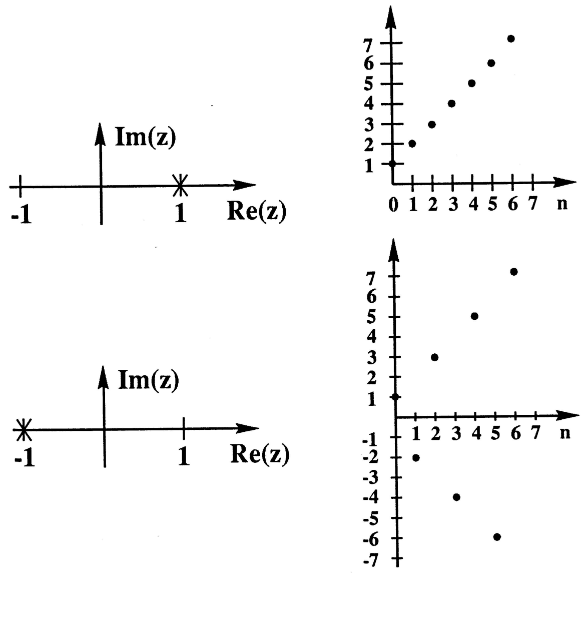
$$\frac{A}{(1-pz^{-1})^2} \overset{ZT^{-1}}{\rightarrow} ?$$

Recall $na^n u(n) \overset{ZT}{\leftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$

$$\frac{A}{(1-pz^{-1})^2} = (A/p)z\left[\frac{pz^{-1}}{(1-pz^{-1})^2}\right] \rightarrow \frac{A}{p}(n+1)p^{n+1}u(n+1)$$

$$\frac{A}{p}(n+1)p^{n+1}u(n+1) = A(n+1)p^n u(n)$$

Example (A=1)



- Similar results are obtained with complex conjugate poles that have multiplicity 2
- In general, repeating a pole with multiplicity m results in multiplication of the signal obtained with multiplicity 1 by a polynomial in n with degree $m-1$

Example

$$\frac{A}{(1-pz^{-1})^3} \overset{ZT^{-1}}{\rightarrow} A(n+1)(n+2)p^n u(n)$$

Stability Considerations

- A system is BIBO stable if every bounded input produces a bounded output
- A DT LTI system is BIBO stable $\Leftrightarrow \sum_n |h(n)| < \infty$
- $\sum_n |h(n)| < \infty \Leftrightarrow H(z)$ converges on the unit circle
- A causal DT LTI system is BIBO stable $\Leftrightarrow$ all poles of H(z) are strictly inside the unit circle

Stability and the general form of the response

1. real pole with multiplicity 1

   $Ap^n u(n)$ is bounded if $|p| \leq 1$

2. complex conjugate pair of poles with multiplicity 1

   $2|A||p|^n \cos(\angle pn + \angle A)u(n)$ is bounded if $p| \leq 1$

3. real pole with multiplicity 2

   $A(n+1)p^n u(n)$ is bounded if $|p| < 1$

Chapter 6

Discrete Fourier Transform (DFT)

6.1 Chapter Outline

In this chapter, we will discuss:

1. Derivation of the DFT
2. DTFT Properties and Pairs
3. Spectral Analysis via the DFT
4. Fast Fourier Transform (FFT) Algorithm
5. Periodic Convolution

6.2 Derivation of the DFT

Summary of Spectral Representations

Signal Type	Transform	Frequency Domain
CT, Periodic	CT Fourier Series	Discrete
CT, Aperiodic	CT Fourier Transform	Continuous
DT, Aperiodic	DT Fourier Transform	Continuous
DT, Periodic	Discrete Fourier Transform	Discrete

Computation of the DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

In order to compute $X(e^{j\omega})$, we must do two things:

1. Truncate summation so that it ranges over finite limits
2. Discretize $\omega = \omega_k$

Suppose we choose

1. $\widetilde{x}(n) = \begin{cases} x(n), & 0 \leq n \leq N-1 \\ 0, & \text{else} \end{cases}$
2. $\omega_k = 2\pi k/N, \quad k = 0, ..., N-1$

then we obtain

$$\widetilde{X}(k) \equiv \sum_{n=0}^{N-1} \widetilde{x}(n)e^{-j2\pi kn/N}$$

This is the forward DFT. To obtain the inverse DFT, we could discretize

the inverse DTFT:

$x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

$\omega \rightarrow \omega_k = \frac{2\pi k}{N}$

$\int_0^{2\pi} d\omega \rightarrow \frac{2\pi}{N}\sum_{k=0}^{N-1}$

$X(e^{j\omega}) \rightarrow \widetilde{X}(k)$

$e^{j\omega n} \rightarrow e^{j2\pi k/N}$

This results in

$$x(n) \approx \frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}(k)e^{j2\pi kn/N}$$

- Note approximation sign
  1. truncated x(n) so $\widetilde{X}(k) \approx X(e^{j2\pi k/N})$
  2. approximated integral by a sum

Alternate approach

- Use orthogonality of complex exponential signals
- Consider again

$$\widetilde{X}(k) = \sum_{n=0}^{N-1} \widetilde{x}(n) e^{-j2\pi kn/N}$$

- For fixed $0 \leq m \leq N-1$, multiply both sides by $e^{j2\pi km/N}$ and sum over k

$$\begin{aligned}
\sum_{k=0}^{N-1} \widetilde{X}(k) e^{j2\pi km/N} &= \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} \widetilde{x}(n) e^{-j2\pi kn/N} \right] e^{j2\pi km/N} \\
&= \sum_{n=0}^{N-1} \widetilde{x}(n) \sum_{k=0}^{N-1} e^{-j2\pi k(n-m)/N} \\
&= \sum_{n=0}^{N-1} \widetilde{x}(n) \left[\frac{1 - e^{-j2\pi(n-m)}}{1 - e^{-j2\pi(n-m)/N}} \right]
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{N-1} \widetilde{X}(k) e^{j2\pi km/N} &= \sum_{n=0}^{N-1} \widetilde{x}(n) N \delta(n-m) \\
&= N \widetilde{x}(m)
\end{aligned}$$

$$\therefore \widetilde{x}(m) = \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{X}(k) e^{j2\pi km/N}$$

Summarizing

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

where we have dropped the tildes

6.3 DFT Properties and Pairs

1. Linearity

$$a_1x_1(n) + a_2x_2(n) \stackrel{DFT}{\leftrightarrow} a_1X_1(k) + a_2X_2(k)$$

2. Shifting

$$x(n - n_0) \stackrel{DFT}{\leftrightarrow} X(k)e^{-j2\pi kn_0/N}$$

3. Modulation

$$x(n)e^{j2\pi k_0 n/N} \stackrel{DFT}{\leftrightarrow} X(k - k_0)$$

4. Reciprocity

$$X(n) \stackrel{DFT}{\leftrightarrow} Nx(-k)$$

5. Parseval's relation

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

6. Initial value

$$\sum_{n=0}^{N-1} x(n) = X(0)$$

7. Periodicity

$$x(n + mN) = x(n) \text{for all integers m}$$

$$X(k + \ell N) = X(k) \text{ for all integers } \ell$$

8. Relation to DTFT of a finite length sequence
   Let $x_0(n) \neq 0$ only for $0 \leq n \leq N - 1$

$$x_0(n) \stackrel{DTFT}{\leftrightarrow} X_0(e^{j\omega})$$

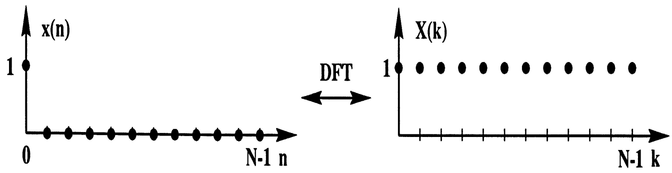
Define $x(n) = \sum_m x_0(n + mN)$

$$x(n) \stackrel{DFT}{\leftrightarrow} X(k)$$

Then $X(k) = X_0(e^{j2\pi k/N})$

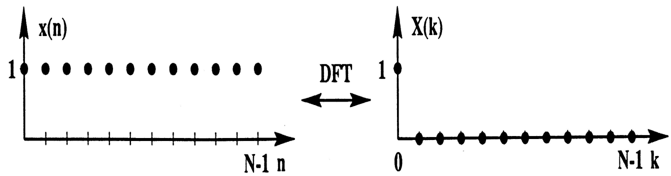
Transform Pairs

1. $x(n) = \delta(n), \quad 0 \leq n \leq N-1$



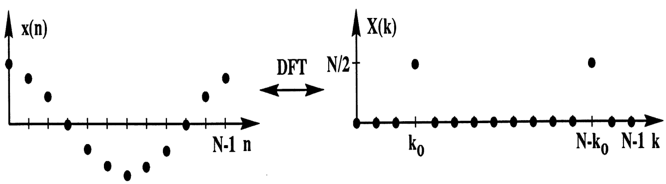
$X(k) = 1, \quad 0 \leq k \leq N-1$ (by relation to DTFT)

2. $x(n) = 1, \quad 0 \leq n \leq N-1$
$X(k) = N\delta k, \quad 0 \leq k \leq N-1$ (by reciprocity)



3. $x(n) = e^{j2\pi k_0 n}, \quad 0 \leq n \leq N-1$
$X(k) = N\delta(k-k_0), \quad 0 \leq k \leq N-1$ (by modulation property)

4. $x(n) = \cos(2\pi k_0 n/N), \quad 0 \leq n \leq N-1$
$$X(k) = \frac{N}{2}\left[\delta(k-k_0) + \delta(k-(N-k_0))\right], \quad 0 \leq k \leq N-1$$



5. $x(n) = \begin{cases} 1, & 0 \leq n \leq M-1, 0 \leq n \leq N-1 \\ 0, & \text{else} \end{cases}$

$$X(k) = e^{j\frac{2\pi k}{N}(M-1)/2}\frac{\sin[2\pi kM/(2N)]}{\sin[2\pi k/(2N)]} \text{ (by relation to DTFT)}$$

6.4 Spectral Analysis Via the DFT

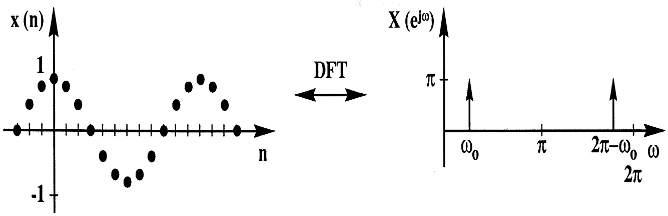
An important application of the DFT is to numerically determine the spectral content of signals. However, the extent to which this is possible is limited by two factors:

1. Truncation of the signal - causes *leakage*
2. Frequency domain sampling - causes *picket fence effect*

Truncation of the Signal

Let $x(n) = \cos(\omega_0 n)$
Recall that

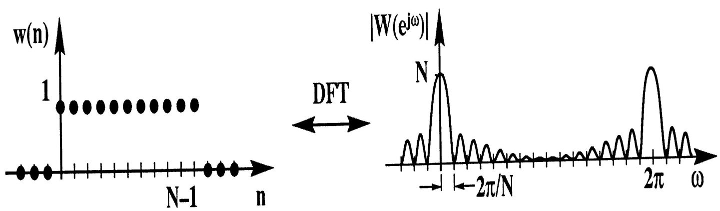
$$X(e^{j\omega}) = \text{rep}_{2\pi}[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$



Also, let $w(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{else} \end{cases}$

Recall $W(e^{j\omega}) = e^{-j\omega(N-1)/2} \dfrac{\sin(\omega N/2)}{\sin(\omega/2)}$

Note that $W(e^{j\omega}) = W(e^{j(\omega + 2\pi m)})$ for all integers m.



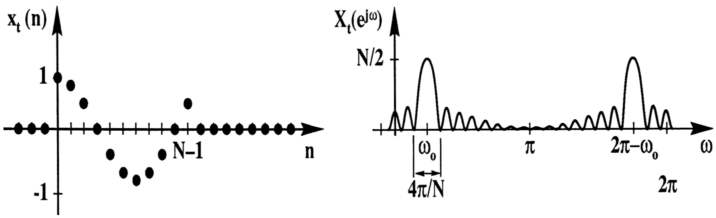
Now let $x_t(n) = x(n)w(n)$ t-truncated

By the product theorem

$$\begin{aligned} X_t(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j(\omega-\mu)})X(e^{j\mu})d\mu \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j(\omega-\mu)})rep_{2\pi}[\pi\delta(\mu-\omega_0)+\pi\delta(\mu+\omega_0)]d\mu \end{aligned}$$

$$\begin{aligned} X_t(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j(\omega-\mu)})[\pi\delta(\mu-\omega_0)+\pi\delta(\mu+\omega_0)]d\mu \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j(\omega-\mu)})\delta(\mu-\omega_0)d\mu \\ &+ \int_{-\pi}^{\pi} W(e^{j(\omega-\mu)})\delta(\mu+\omega_0)d\mu \end{aligned}$$

$X_t(e^{j\omega}) = \frac{1}{2}\left[W(e^{j(\omega-\omega_0)}) + W(e^{j(\omega+\omega_0)})\right]$



Frequency Domain Sampling

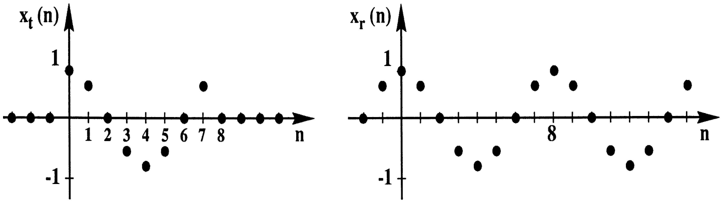
Let

$x_r(n) = \sum_m x_t(n+mN)$ r-replicated

Recall from relation between DFT and DTFT of finite length sequence that

$$\begin{aligned} X_r(k) &= X_t(e^{j2\pi k/N}) \\ &= \frac{1}{2}\left[W(e^{j(2\pi k/N-\omega_0)}) + W(e^{j(2\pi k/N+\omega_0)})\right] \end{aligned}$$

Consider two cases:

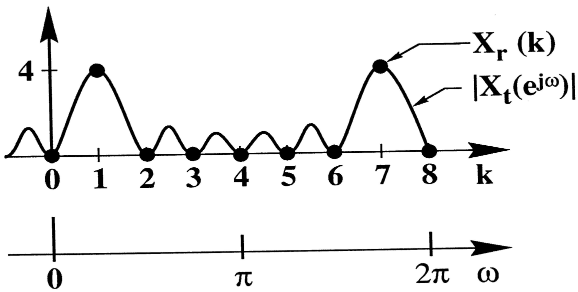


1. $\omega_0 = 2\pi k_0/N$ for some integer k_0

   Example: $k_0 = 1, \quad N = 8, \quad \omega_0 = \pi/4$

   $$X_r(k) = \frac{1}{2\pi}\left[W(e^{j2\pi(k-k_0)/N}) + W(e^{j2\pi(k+k_0)/N})\right]$$

   Example: $k_0 = 1, \quad N = 8$



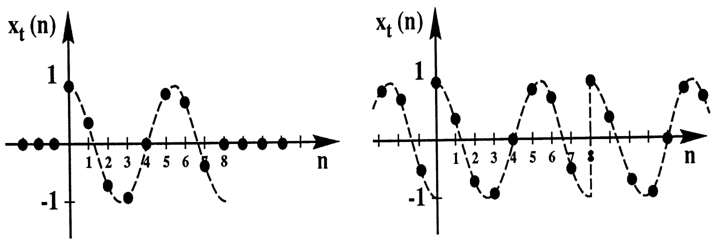
   In this case:

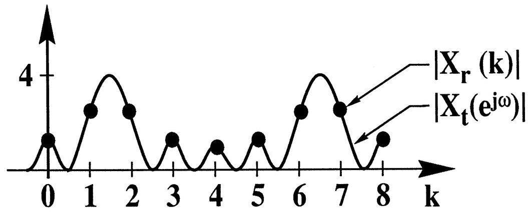
   $$X_r(k) = \frac{N}{2}\left[\delta(k-k_0) + \delta(k-(N-k_0))\right], \quad 0 \le k \le N-1$$

   as shown before. There is no picket fence effect.

2. $\omega_0 = 2\pi k_0/N + \pi/N$ for some integer k_0

   Example: $k_0 = 1, \quad N = 8, \quad \omega_0 = 3\pi/8$





Picket fence effect:

- Spectral peak is midway between sample locations
- Each sidelobe peak occurs at a sample location

1. Both truncation and picket fence effects are reduced by increasing N
2. Sidelobes may be supressed at the expense of a wider mainlobe by using a smoothly tapering window

6.5 Fast Fourier Transform (FFT) Algorithm

The FFT is an *algorithm* for efficient computation of the DFT. It is not a new transform.

Recall

$$X^{(N)}(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, ..., N-1$$

Superscript (N) is used to show the length of the DFT. For each value of k, computation of X(k) requires N complex multiplications and N-1 additions.

Define a *complex operation* (CO) as 1 complex multiplication and 1 complex addition.

Computation of length N DFT then requires approximately N^2 CO's.

To derive an efficient algorithm for computation of the DFT, we employ a divide-and-conquer strategy.

Assume N is even.

$$X^{(N)}(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$\begin{aligned} X^{(N)}(k) &= \sum_{n=0,even}^{N-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0,odd}^{N-1} x(n)e^{-j2\pi kn/N} \\ &= \sum_{m=0}^{N/2-1} x(2m)e^{-j2\pi k(2m)/N} \\ &+ \sum_{m=0}^{N/2-1} x(2m+1)e^{-j2\pi k(2m+1)/N} \end{aligned}$$

$X^{(N)}(k) =$
$\sum_{m=0}^{N/2-1} x(2m)e^{-j2\pi km/(N/2)} + e^{-j2\pi k/N} \sum_{m=0}^{N/2-1} x(2m+1)e^{-j2\pi km/(N/2)}$

Let $x_0(n) = x(2m), \qquad m = 0, ..., N/2-1$

$x_1(n) = x(2m+1), \qquad m = 0, ..., N/2-1$

Now have

$$X^{(N)}(k) = X_0^{(N/2)}(k) + e^{-j2\pi k/N} X_1^{(N/2)}(k), \quad k = 0, ..., N-1$$

Note that $X_0^{(N/2)}(k)$ and $X_1^{(N/2)}(k)$ are both periodic with period N/2, while $e^{-j2\pi k/N}$ is periodic with period N.

Computation

- Direct

  $X^{(N)}(k), \quad k = 0, ..., N-1 \qquad N^2$ CO's

- Decimation by a factor of 2

  $X_0^{(N/2)}(k), \; k = 0, ..., N/2-1 \qquad N^2/4$ CO's

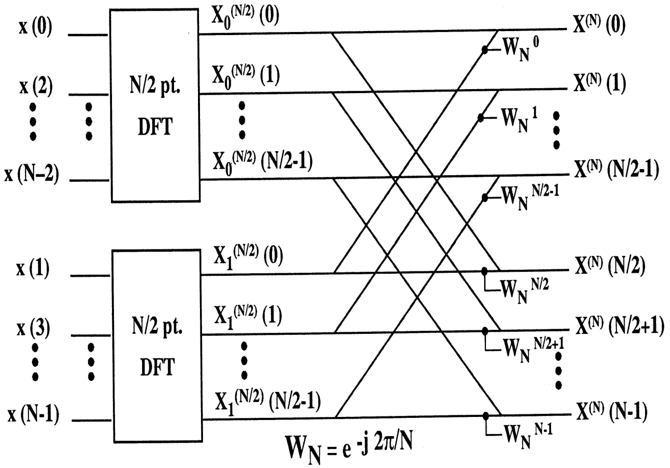
  $X_1^{(N/2)}(k), \; k = 0, ..., N/2-1 \qquad N^2/4$ CO's

  $X^{(N)}(k) = X_0^{(N/2)}(k) + e^{-j2\pi k/N} X_1^{(N/2)}(k), \quad k = 0, ..., N-1$ N CO's

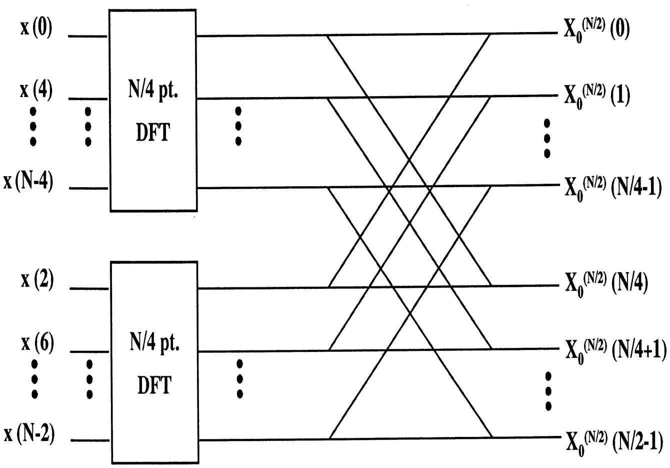
Total $N^2/2 + N$ CO's

For large N, we have nearly halved computation.

Consider a signal flow diagram of what we have done so far:



If N is even,we can repeat the idea with each N/2 pt. DFT:



If $N = 2^M$, we repeat the process M times reulting in M stages.

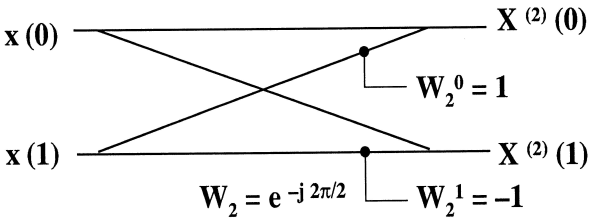
The first stage consists of 2 point DFT's:

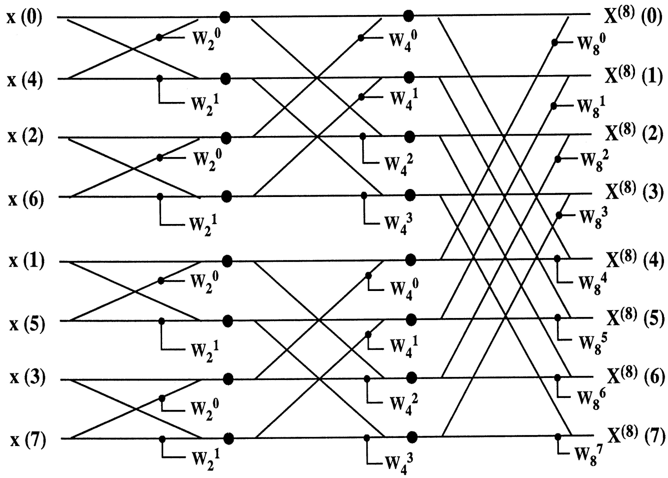
$$X^{(2)}(k) = \sum_{n=0}^{1} x(n)e^{-j2\pi kn/2}$$

$$X^{(2)}(0) = x(0) + x(1)$$

$$X^{(2)}(1) = x(0) - x(1)$$

Flow diagram of 2 pt. DFT Full example for N=8(M=3)





Ordering of Input Data

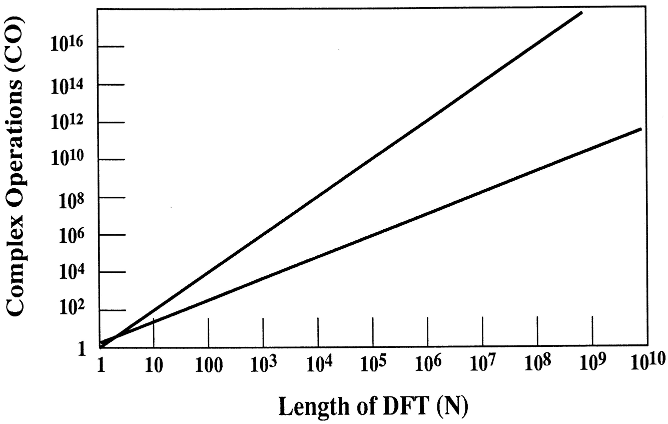
Normal Order		Bit Reversed Order	
Decimal	Binary	Binary	Decimal
0	0 0 0	0 0 0	0
1	0 0 1	1 0 0	4
2	0 1 0	0 1 0	2
3	0 1 1	1 1 0	6
4	1 0 0	0 0 1	1
5	1 0 1	1 0 1	5
6	1 1 0	0 1 1	3
7	1 1 1	1 1 1	7

Computation ($N = 2^M$)

$M = \log_2 N$ stages

N CO's/stage

Total: $N\log_2 N$ CO's



Comments

- The algorithm we derived is the decimation-in-time radix 2 FFT

    1. input in bit-reversed order

2. in-place computation
3. output in normal order

- The dual of the above algorithm is the decimation-in-frequency radix 2 FFT
  1. input in normal order
  2. in-place computation
  3. output in normal order
- The same approaches may be used to derive either decimation-in-time or decimation-in-frequency mixed radix FFT algorithms for any N which is a composite number
- All FFT algorithms are based on composite N and require O(N log N) computation
- The DFT of a length N real signal can be expressed in terms of a length N/2 DFT

6.6 Periodic Convolution

We developed the DFT as a computable spectral representation for DT signals. However, the existence of a very efficient algorithm for calculating it suggests another application. Consider the filtering of a length N signal x(n) with an FIR filter containing M coefficients, where $M << N$

$$y(n) = \sum_{m=0}^{M-1} h(m)x(n-m)$$

Computation of each output point requires M multiplications and M-1 additions.

Based on the notion that convolution in the time domain corresponds to multiplication in the frequency domain, we perform the following computations:

- Compute $X^{(n)}(k) \quad N \log_2 N$ CO's
- Extend h(n) with zeros to length N and compute $H^{(N)}(k) \quad N \log_2 N$ CO's
- Multiply the DFT's

- $Y^{(N)}(k) = H^{(N)}(k)X^{(N)}(k)$ N complex mutliplications
- Compute inverse DFT

  $\widetilde{y}(n) = DFT^{-1}\left\{Y^{(N)}(k)\right\} N\log_2 N$ CO's

  Total: $3N\log_2 N$ CO's + N complex multiplications

  The computation/output point is 3 $\log_2$ N CO's+ 1 complex multiplication.

How Periodic Convolution Arises

- Consider inverse DFT of product $Y[k] = H[k]X[k]$

  $$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} H[k]X[k]e^{j2\pi kn/N}$$

- Substitute for X[k] in terms of x[n]

  $$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} H[k]\left\{\sum_{m=0}^{N-1} x[m]e^{-j2\pi km/N}\right\}e^{j2\pi kn/N}$$

- Interchange order of summations

  $$\begin{aligned} y(n) &= \sum_{m=0}^{N-1} x[m]\left\{\frac{1}{N}\sum_{k=0}^{N-1} H[k]e^{-j2\pi k(n-m)/N}\right\} \\ &= \sum_{m=0}^{N-1} x[m]h[n-m] \end{aligned}$$

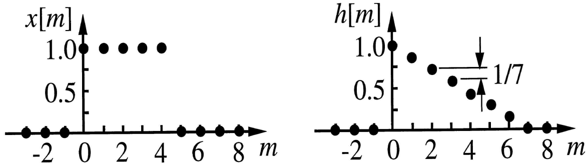
- Note periodicity of sequence h[n]

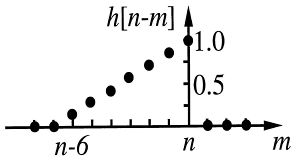
  $h[n-m] = h[(n-m) \bmod N]$

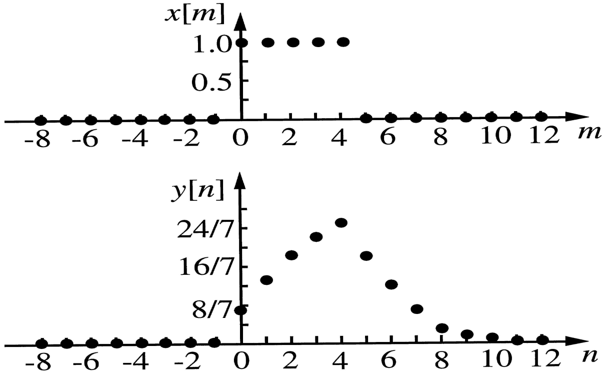
Periodic vs. Aperiodic Convolution

Aperiodic convolution

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[n-m]x[m]$$

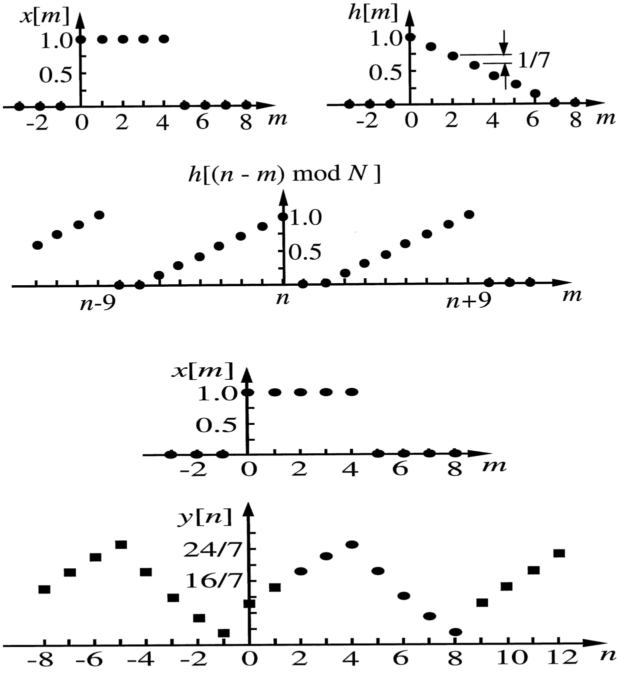






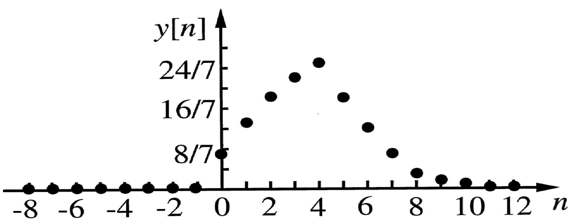
Periodic convolution (N=9)

$$y[n] = h[n] \otimes x[n] = \sum_{m=0}^{N-1} h[(n-m)modN]x[m]$$

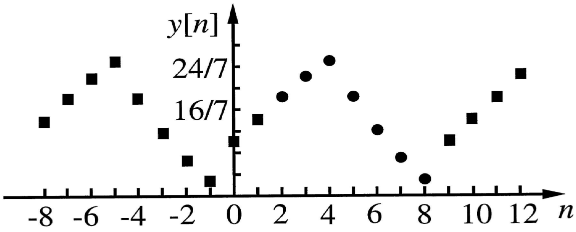


Comparison

Aperiodic convolution

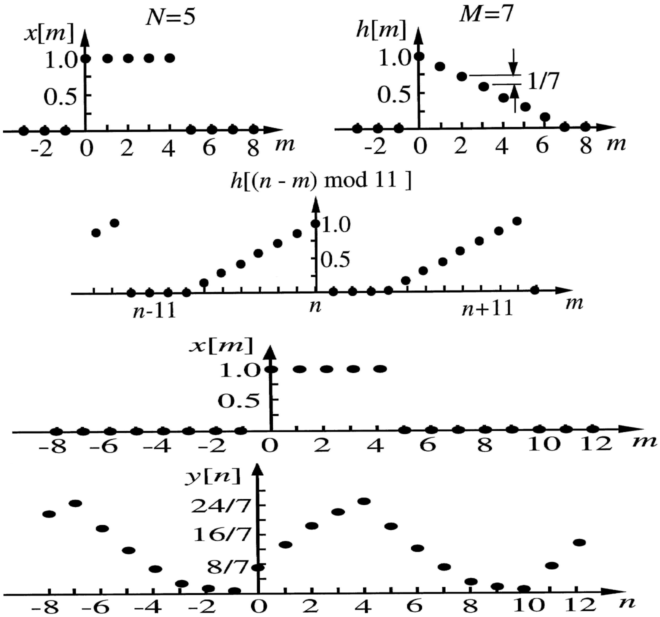


Periodic convolution ($N = 9$)



Zero Padding to Match Aperiodic Convolution

Suppose x[n] has length N and h[n] has length M. Periodic convolution with length M+N-1 will match aperiodic result



Chapter 7

Digital Filter Design

7.1 Chapter Outline

In this chapter, we will discuss:

1. Overview of filter design problem
2. Finite impulse response filter design
3. Infinite impulse response filter design

7.2 Overview

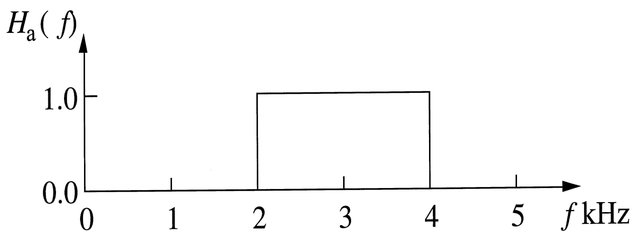
- Filter design problem consists of three tasks
    1. Specification - What frequency response or other characteristics of the filter are desired?
    2. Approximation - What are the coefficients or, equivalently, poles and zeros of the filter that will approximate the desired characteristics?
    3. Realization - How will the filter be implemented?
- Consider a simple example to illustrate these tasks.

Filter Design Example

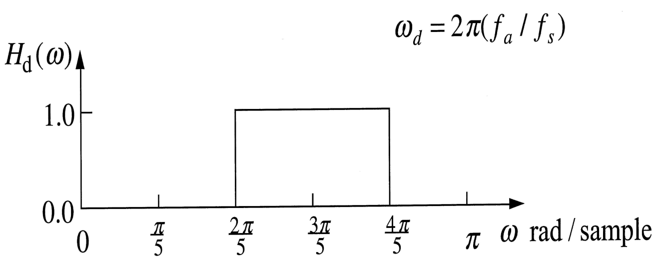
- An analog signal contains frequency components ranging from DC (0 Hz) to 5 kHz
- Design a digital system to remove all but the frequencies in the range 2-4 kHz
- Assume signal will be sampled at 10 kHz rate

Specification

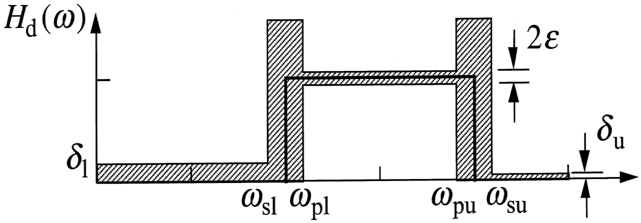
- Ideal analog filter



- Ideal digital filter



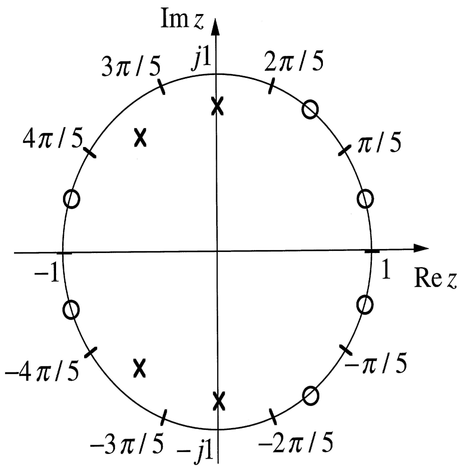
Filter Tolerance Specification



ω_{pl}, ω_{pu} - lower and upper passband edges
ω_{sl}, ω_{su} - lower and upper stopband edges
δ_l, δ_u - lower and upper stopband ripple
ϵ - passband ripple

Approximation

- Design by positioning poles and zeros (PZ plot design)

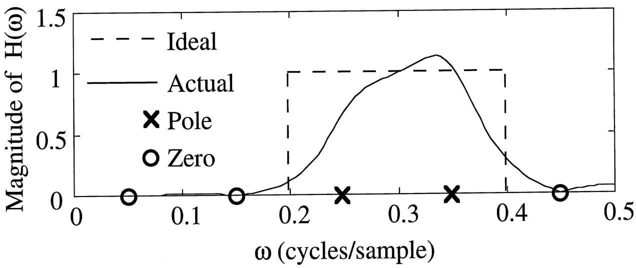


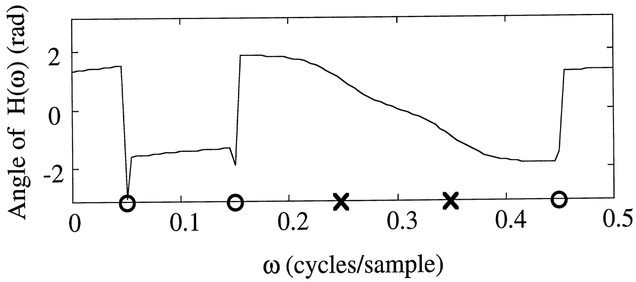
- Transfer function for PZ plot filter

$$H(z) = K \frac{\left(z - e^{j\pi/10}\right)\left(z - e^{j3\pi/10}\right)\left(z - e^{j9\pi/10}\right)}{\left(z - 0.8e^{j5\pi/10}\right)\left(z - 0.8e^{j7\pi/10}\right)} \times \frac{\left(z - e^{-j\pi/10}\right)\left(z - e^{-j3\pi/10}\right)\left(z - e^{-j9\pi/10}\right)}{\left(z - 0.8e^{-j5\pi/10}\right)\left(z - 0.8e^{-j7\pi/10}\right)}$$

  Choose constant K to yield unity magnitude response at center of passband

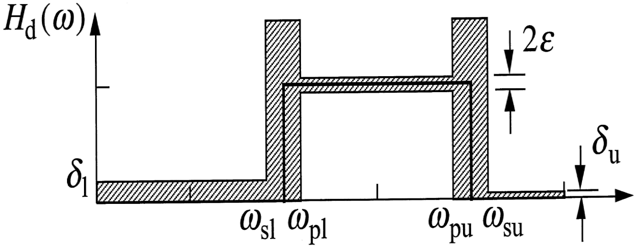
- Frequency response of PZ plot filter





- Comments on PZ plot design

  1. Passband asymmetry is due to extra zeros in lower stopband
  2. Phase is highly nonlinear
  3. Zeros on unit circle drive magnitude response to zero and result in phase discontinuities
  4. Pole-zero plot yields intuition about filter behavior, but does not suggest how to meet design specifications:



Realization

- Cascade form of transfer function

$$H(z) = K \frac{\left(z - e^{j\pi/10}\right)\left(z - e^{j3\pi/10}\right)\left(z - e^{j9\pi/10}\right)}{\left(z - 0.8e^{j5\pi/10}\right)\left(z - 0.8e^{j7\pi/10}\right)} \times \frac{\left(z - e^{-j\pi/10}\right)\left(z - e^{-j3\pi/10}\right)\left(z - e^{-j9\pi/10}\right)}{\left(z - 0.8e^{-j5\pi/10}\right)\left(z - 0.8e^{-j7\pi/10}\right)}$$

Cascade Form Realization

- Cascade form in second order sections with real-valued coefficients

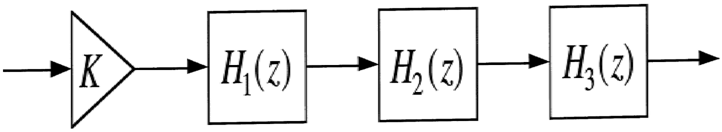
$$H(z) = K[z^2 - 2\cos(\pi/10)z + 1] \times \left[\frac{z^2 - 2\cos(3\pi/10)z + 1}{z^2 - 1.6\cos(5\pi/10)z + 0.64}\right] \times \left[\frac{z^2 - 2\cos(9\pi/10)z + 1}{z^2 - 1.6\cos(7\pi/10)z + 0.64}\right]$$

- Convert to negative powers of z

$$H(z) = K[1 - 2\cos(\pi/10)z^{-1} + z^{-2}] \times \left[\frac{1 - 2\cos(3\pi/10)z^{-1} + z^{-2}}{1 - 1.6\cos(5\pi/10)z^{-1} + 0.64z^{-2}}\right] \left[\frac{1 - 2\cos(9\pi/10)z^{-1} + z^{-2}}{1 - 1.6\cos(7\pi/10)z^{-1} + 0.64z^{-2}}\right]$$

(Ignoring overall time advance of 2 sample units)

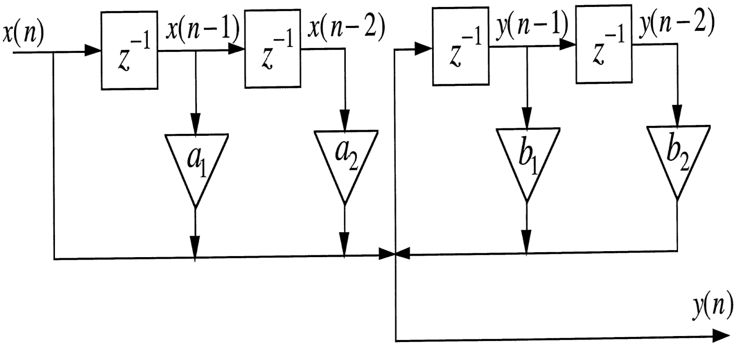
- Overall System



- Second order stages

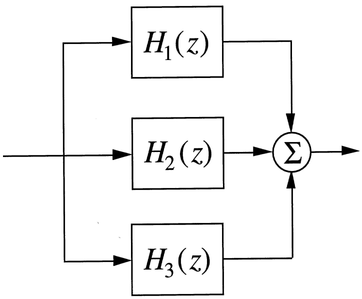
$$H_i(z) = \frac{1 + a_{1i}z^{-1} + a_{2i}z^{-2}}{1 - b_{1i}z^{-1} - b_{2i}z^{-2}}, \ i = 1, 2, 3$$

$$y_i[n] = x_i[n] + a_{1i}x_i[n-1] + a_{2i}x_i[n-2] + b_{1i}y_i[n-1] + b_{2i}y_i[n-2]$$



Parallel Form Realization

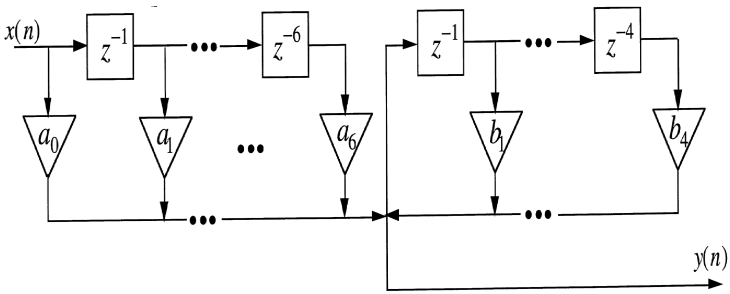
- Expand transfer function in partial fractions
  $H(z) =$
  $$d_0 + d_1z^{-1} + d_2z^{-2} + \frac{a_{01} + a_{11}z^{-1}}{1 - b_{11}z^{-1} - b_{21}z^{-2}} + \frac{a_{02} + a_{12}z^{-1}}{1 - b_{12}z^{-1} - b_{22}z^{-2}}$$



Direct Form Realization

Multiply out all factors in numerator and denominator of transfer function

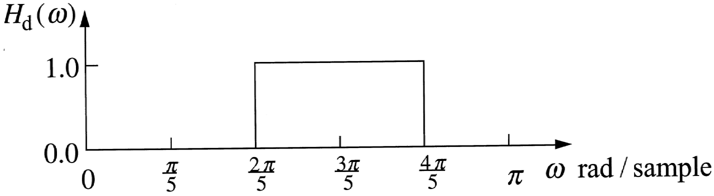
$$H(z) = \frac{a_0 + a_1 z^{-1} + ... + a_6 z^{-6}}{1 - b_1 z^{-1} - ... - b_4 z^{-4}}$$



7.3 Finite Impulse Response Filter Design

Ideal Impulse Response

- Consider same ideal frequency response as before



- Inverse DTFT

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega$$

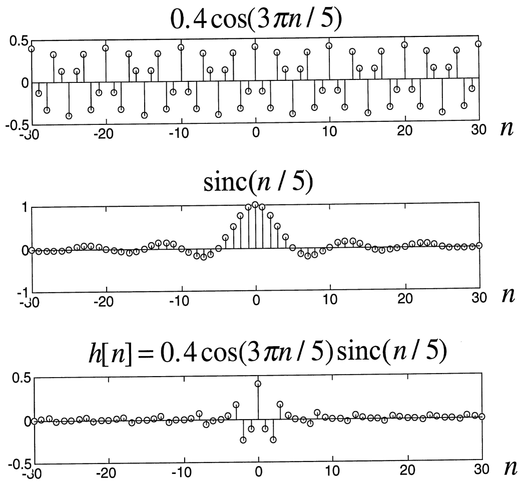
$$h[n] = \frac{1}{2\pi}\left\{\int_{-4\pi/5}^{-2\pi/5} e^{j\omega n}d\omega + \int_{2\pi/5}^{4\pi/5} e^{j\omega n}d\omega\right\}$$

$$h[n] = \frac{1}{j2\pi n}\left\{\left[e^{-j2\pi n/5} - e^{-j4\pi n/5}\right] + \left[e^{j4\pi n/5} - e^{-j2\pi n/5}\right]\right\}$$

$$h[n] = \frac{1}{j2\pi n}\left\{e^{-j3\pi n/5}\left[e^{j\pi n/5} - e^{-j\pi n/5}\right] + e^{j3\pi n/5}\left[e^{j\pi n/5} - e^{-j\pi n/5}\right]\right\}$$

$$h[n] = \left\{e^{-j3\pi n/5}\frac{1}{5}\text{sinc}\left(\frac{n}{5}\right) + e^{j3\pi n/5}\frac{1}{5}\text{sinc}\left(\frac{n}{5}\right)\right\}$$

$$h[n] = 0.4\cos(3\pi n/5)\text{sinc}(n/5)$$



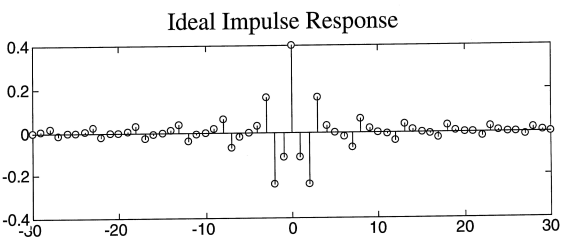
Approximation of Desired Filter Impulse Response by Truncation

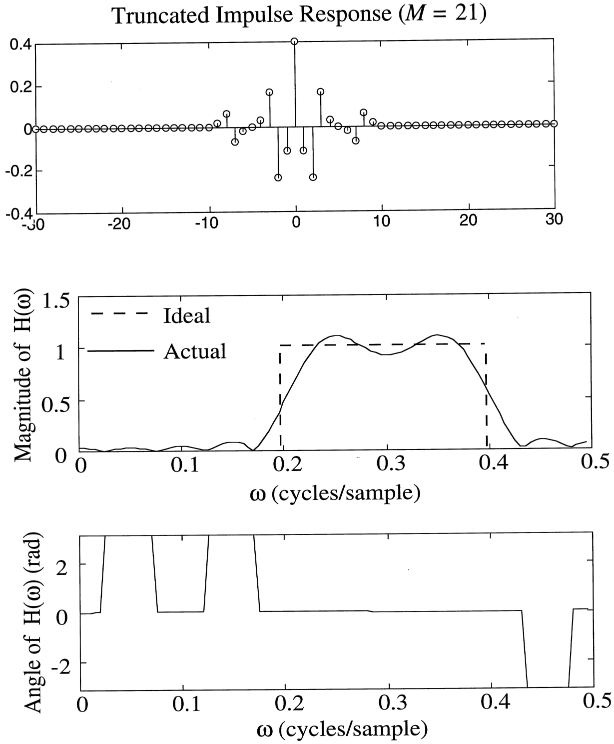
- FIR filter equation

$$y[n] = \sum_{m=0}^{M-1} a_m x[n-m]$$

- Filter coefficients (assuming M is odd)
  $a_m = h_{ideal}[m-(M-1)/2], \quad m = 0, ..., M-1$

Truncation of Impulse Response



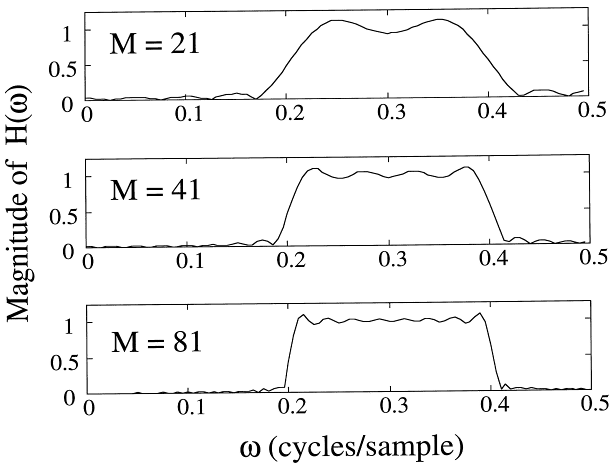


Frequency Response of Truncated Filter

Analysis of Truncation

- ideal infinite impulse response - $h_{ideal}[n]$
- Actual, finite impulse response - $h_{actual}[n]$
- Window sequence - w[n]
  $$w[n] = \begin{cases} 1 & , |n| \leq (M-1)/2 \\ 0 & , \text{else} \end{cases}$$
- Relation between ideal and actual impulse responses
  $h_{actual}[n] = h_{ideal}[n]w[n]$
  $$H_{actual}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{ideal}(\lambda) W(\omega - \lambda) d\lambda$$

Effect of Filter Length



DTFT of Rectangular Window

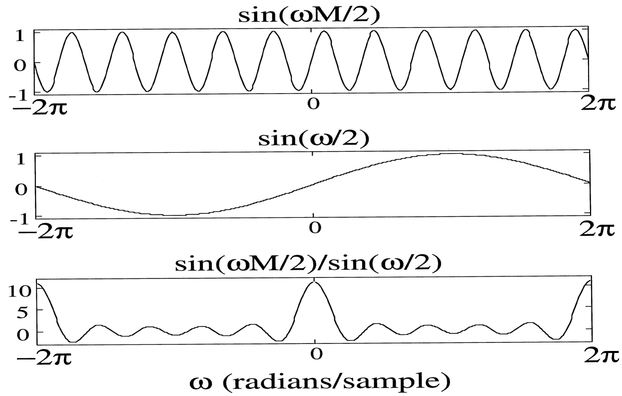
$$W(\omega) = \sum_{n=-(M-1)/2}^{n=(M-1)/2} e^{-j\omega n}$$

let m=n+(M-1)/2

$$= \sum_{m=0}^{M-1} e^{-j\omega[m-(M-1)/2]}$$

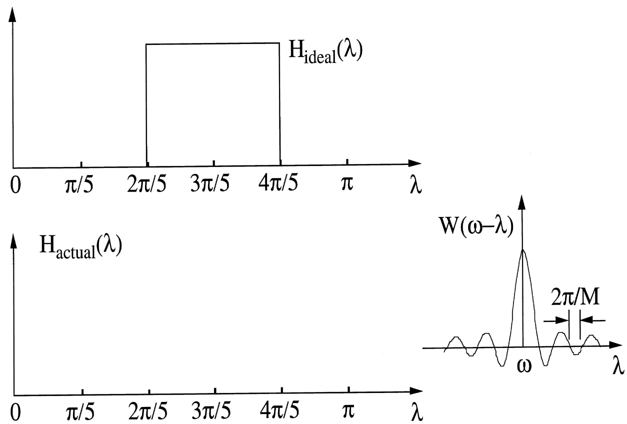
$$= e^{j\omega(M-1)/2} \frac{1-e^{j\omega M}}{1-e^{j\omega}}$$

$$= \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$



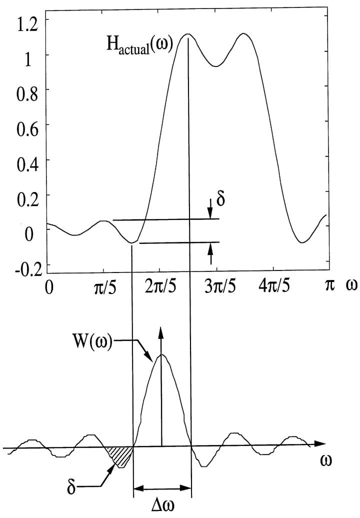
Graphical View of Convolution

$$H_{actual}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{ideal}(\lambda)W(\omega-\lambda)d\lambda$$



Relation between Window Attributes and Filter Frequency Response

Parameter	$W(\omega)$	$H_{actual}(\omega)$
$\Delta\omega$	Mainlobe width	Transition bandwidth
δ	Area of first sidelobe	Passband and stopband ripple



Design of FIR Filters by Windowing

- Relation between ideal and actual impulse responses

$$h_{actual}[n] = h_{ideal[n]}w[n]$$

- Choose window sequence w[n] for which DTFT has
    1. minimum mainlobe width
    2. minimum sidelobe area
- Kaiser window is the best choice
    1. based on optimal prolate spheroidal wavefunctions
    2. contains a parameter that permits tradeoff between mainlobe width and sidelobe area

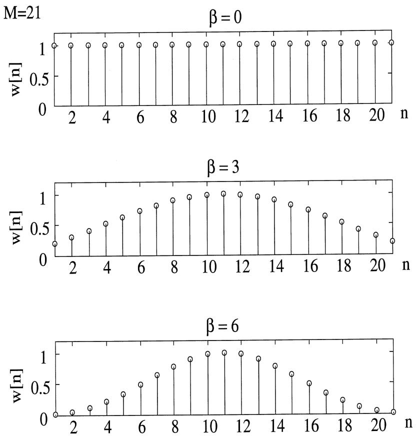
Kaiser Window

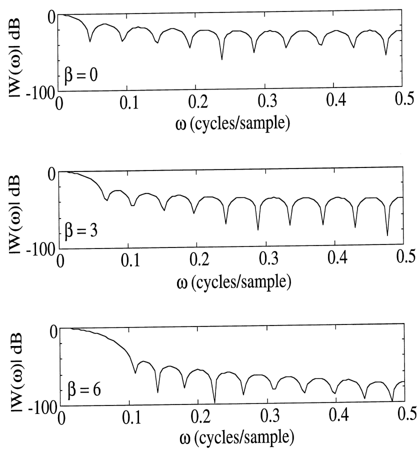
$$w[n] = \begin{cases} \frac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \leq N \leq M-1 \\ 0, & \text{else} \end{cases}$$

$\alpha = (M-1)/2$

$I_0(.)$ zeroth-order modified Bessel function of the first kind

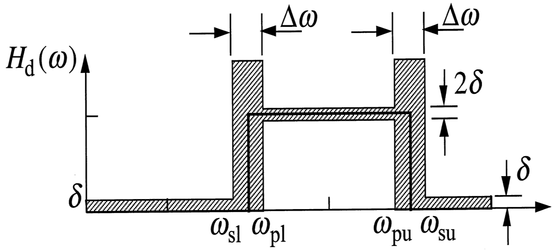
Effect of Parameter β





Filter Design Procedure for Kaiser Window

1. Choose stopband and passband ripple and transition bandwidth

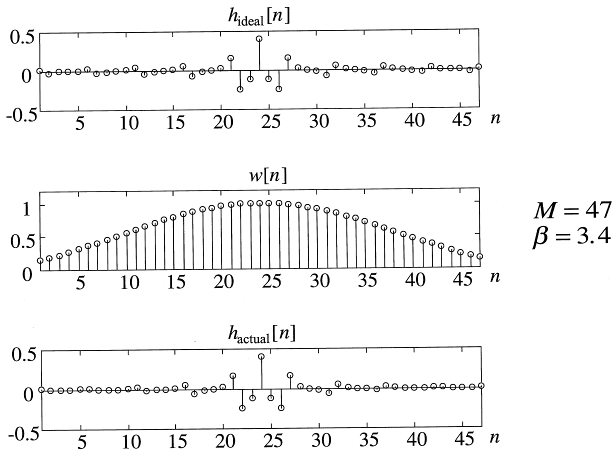


2. Determine paramter β
   $A = -20\log_{10}\delta$
   $$\beta = \begin{cases} 0.112(A-8.7) & , \quad A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & , 21 \leq A \leq 50 \\ 0.0 & , A < 21 \end{cases}$$

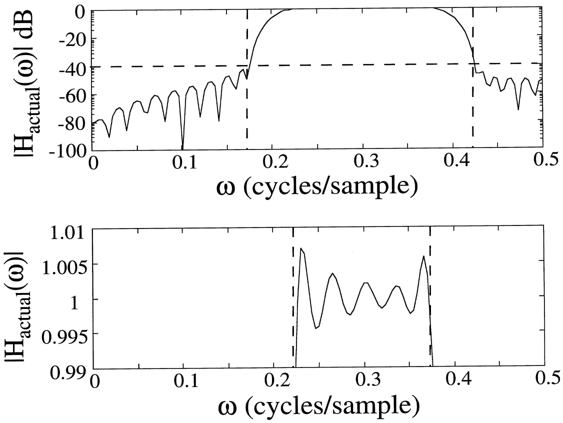
3. Determine filter length M
   $M = \frac{A-8}{2.285\Delta\omega} + 1$

4. Example
   $\Delta\omega = 0.1\pi$ radians/sample= 0.05 cycles/sample
   $\delta = 0.01$
   $\beta = 3.4$
   $M = 45.6$

Filter Designed with Kaiser Window



Kaiser Filter Frequency Response



$M = 47$
$\beta = 3.4$
$\delta = 0.01 = -40$ dB
$\Delta\omega = 0.05$
$\omega_{sl} = 0.175$
$\omega_{pl} = 0.225$
$\omega_{pu} = 0.375$
$\omega_{su} = 0.425$

Optimal FIR Filter Design

- Although the Kaiser window has certain optimal properties, filters designed using this window are not optimal
- Consider the design of filters to minimize integral mean-squared frequency error
- Problem:
  Choose $h_{actual}[n] = -(M-1)/2, ..., (M-1)/2$ to minimize
  $E = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{actual}(\omega) - H_{ideal}(\omega)|^2 d\omega$

Minimum Mean-Squared Error Design

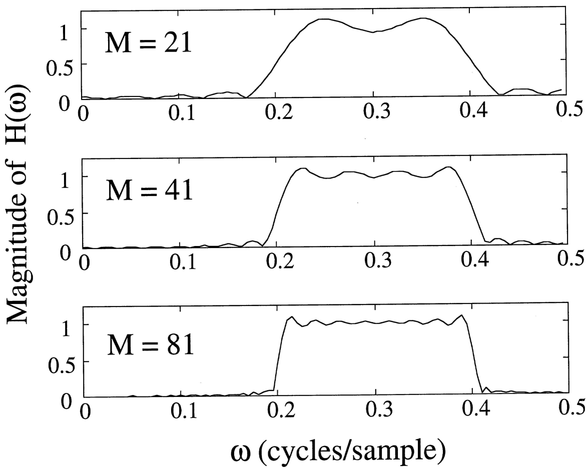
- Using Parseval's relation, we obtain a direct solution to this problem

$$\begin{aligned} E &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{actual}(\omega) - H_{ideal}(\omega)|^2 d\omega \\ &= \sum_{n=-\infty}^{\infty} |h_{actual}[n] - h_{ideal}[n]|^2 \\ &= \sum_{n=-(M-1)/2}^{(M-1)/2} |h_{actual}[n] - h_{ideal}[n]|^2 \\ &+ \sum_{n=-\infty}^{-(M+1)/2} |h_{ideal}[n]|^2 + \sum_{n=(M+1)/2}^{\infty} |h_{ideal}[n]|^2 \end{aligned}$$

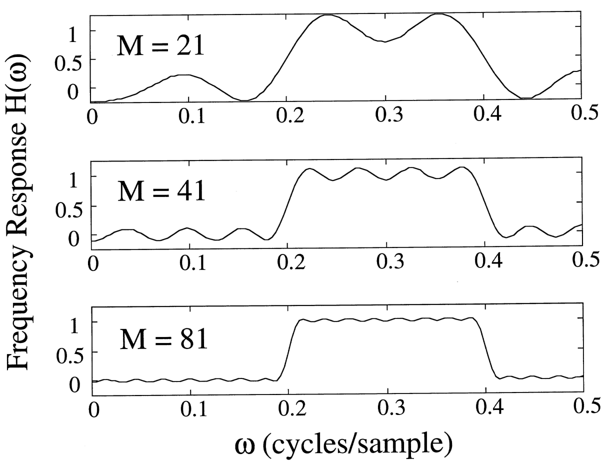
- Solution
  Set $h_{actual}[n] = h_{ideal}[n],\ n = -(M-1)/2, ..., (M-1)/2$
  Minimum error is given by
  $E = \sum_{n=-\infty}^{-(M+1)/2} |h_{ideal}[n]|^2 + \sum_{n=(M+1)/2}^{\infty} |h_{ideal}[n]|^2$
- This is the truncation of the ideal impulse response
- We conclude that original criteria of minimizing mean-squared error was not a good choice
- What was the problem with truncating the ideal impulse response? (See figure on following page)

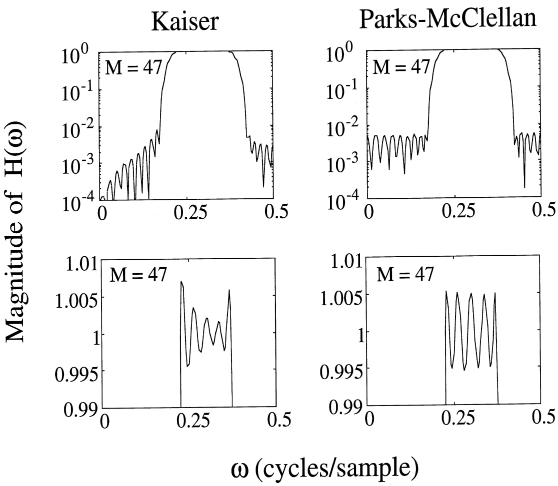
Minimax (Equi-ripple) Filter Design

- No matter how large we pick M for the truncated impulse response, we cannot reduce the peak ripple
- Also, the ripple is concentrated near the band edges
- Rather than focussing on the integral of the error, let's consider the maximum error
- Parks and McClellan developed a method for design of FIR filters based on minimizing the maximum of the weighted frequency domain error



Effect of Filter Length: Parks-McClellan Equiripple Filters





7.4 Infinite Impulse Response Filter Design

IIR vs. FIR Filters

- FIR Filter Equation
  $y[n] = a_0x[n] + a_1x[n-1] + ... + a_{M-1}x[n-(M-1)]$
- IIR Filter Equation
  $y[n] = a_0x[n] + a_1x[n-1] + ...a_{M-1}x[n-(M-1)] - b_1y[n-1] - b_2y[n-2] - ... - b_Ny[n-N]$
- IIR filters can generally achieve given desired response with less computation than FIR filters
- It is easier to approximate arbritrary frequency response characteristics with FIR filters, including exactly linear phase

Infinite Impulse Response (IIR) Filter Design: Overview

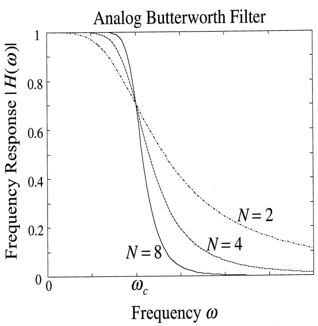
- IIR digital filter designs are based on established methods for designing analog filters
- Approach is generally imited to frequency selective filters with ideal passband/stopband characteristics
- Basic filter type is low pass

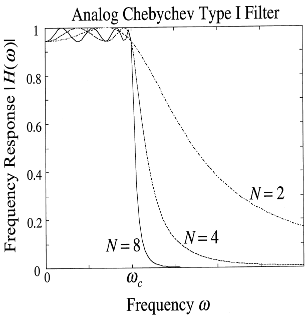
- Achieve highpass or bandpass via transformations
- Achieve multiple stop/pass bands by combining multiple filters with single pass band

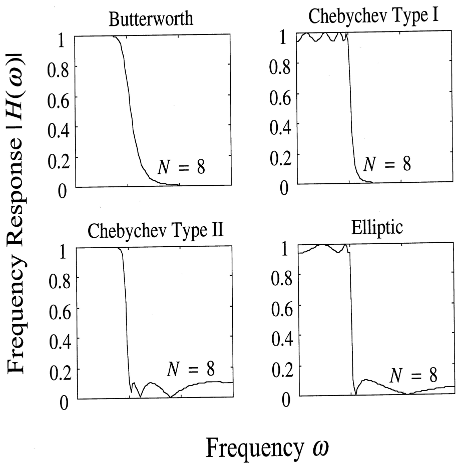
IIR Filter Design Steps

- Choose prototype analog filter family
    1. Butterworth
    2. Chebychev Type I or II
    3. Elliptic
- Choose analog-digital transformation method
    1. Impulse Invariance
    2. Bilinear Transformation
- Transform digital filter specifications to equivalent analog filter specifications
- Design analog filter
- Transform analog filter to digital filter
- Perform frequency transformation to achieve highpass or bandpass filter, if desired

Prototype Filter Types







Transformation via Impulse Invariance

- Sample impulse response of analog filter

  $h_d[n] = T h_a(nT)$

  $$H_d(\omega) = \sum_k H_a(f - k/T)\Big|_{f = \frac{\omega}{2\pi T}}$$

  Note that aliasing may occur

Impulse Invariance

- Implementation of digital filter

1. Partial fraction expansion of analog transfer function (assuming all poles have multiplicity 1)

   $$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

2. Inverse Laplace transform

   $$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

3. Sample Impulse response

   $$\begin{aligned} h_d[n] = Th_a(nT) &= T \sum_{k=1}^{N} A_k e^{s_k nT} u(nT) \\ &= T \sum_{k=1}^{N} A_k p_k^n u[n], \quad p_k = e^{s_k T} \end{aligned}$$

4. Take Z transform

   $$H_d(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - p_k z^{-1}}$$

5. Combine terms

   $$H_d(z) = \frac{a_0 + a_1 z^{-1} + ...a_{M-1} z^{-(M-1)}}{1 + b_1 z^{-1} + b_2 z^{-2} + ...b_N z^{-N}}$$

6. Corresponding filter equation

   $y[n] = a_o x[n] + a_1 x[n-1] + ...a_{M-1} x[n-(M-1)] - b_1 y[n-1] - b_2 y[n-2] - ... - b_N y[n-N]$

Impulse Invariance Example

- Design a second order ideal low pass digital filter with cutoff at $\omega_d = \pi/5$ radians/sample (Assume T=1)

  1. Analog cutoff frequency

     $f_c = \frac{\omega_d}{2\pi T} = 0.1Hz$

     $\omega_c = 2\pi f_c = \pi/5$ rad/sec

- Use Butterworth analog prototype

$$H_a(s) = \frac{0.3948}{[s - (-0.4443 + j0.4443)][s - (-0.4443 - j0.4443)]}$$

- Apply partial fraction expansion

$$H_a(s) = \frac{-j0.4443}{[s-(-0.4443+j0.4443)]} + \frac{j0.4443}{[s-(-0.4443-j0.4443)]}$$

- Compute inverse Laplace Transform

$$h_a(t) = [-j0.4443e^{(-0.4443+j0.4443)t} + j0.4443e^{(-0.4443-j0.4443)t}]u(t)$$

- Sample impulse response

$$\begin{aligned} h_d[n] = & \Big[-j0.4443e^{(-0.4443+j0.4443)n} \\ & +j0.4443e^{(-0.4443-j0.4443)n}\Big]\, u[n] \\ & = \big[-j0.4443(0.641e^{j0.4443})^n \\ & +j0.4443(0.641e^{-j0.4443})^n\big]\, u[n] \end{aligned}$$

- Compute Z transform

$$\begin{aligned} H_d(z) &= \frac{-j0.4443}{1-0.6413e^{j0.4443}z^{-1}} + \frac{j0.4443}{1-0.6413e^{-j0.4443}z^{-1}} \\ &= \frac{0.2450z^{-1}}{1-1.1581z^{-1}+0.4113z^{-2}} \end{aligned}$$

- Combine terms

$$H_d(z) = \frac{0.2450z^{-1}}{(1-0.6413e^{j0.4443}z^{-1})(1-0.6413e^{-j0.4443}z^{-1})}$$

- Find difference equation

$$y[n] = 0.2450x[n-1] + 1.1581y[n-1] - 0.4113y[n-2]$$

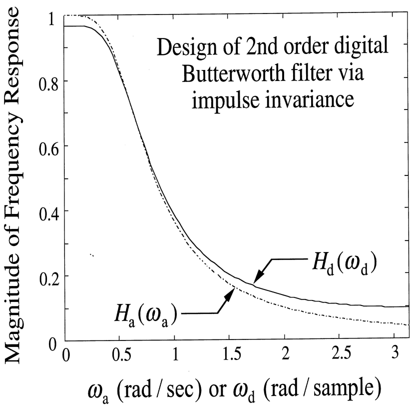
Impulse Invariance Method-Summary

- Preserves impulse response and shape of frequency response, if there is no aliasing

- Desired transition bandwidths map directly between digital and analog frequnecy domains

- Passband and stopband ripple specifications are identical for both digital and analog filter, assuming that there is no aliasing

- The final digital filter design is independent of the sampling interval parameter T
- Poles in analog filter map directly to poles in digital filter via transformation $p_k = e^{s_k T}$
- There is no such relation between the zeros in the two filters
- Gain at DC in a digital filter may not equal unity, since sampled impulse response may only approximately sum to 1



Bilinear Transformation Method

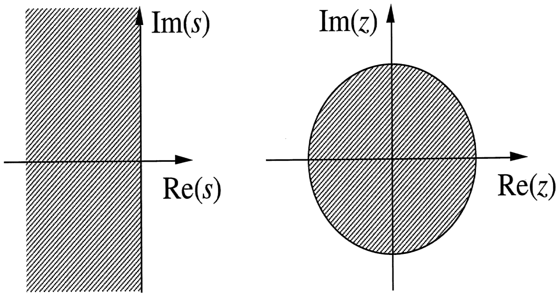
- Mapping between s and z

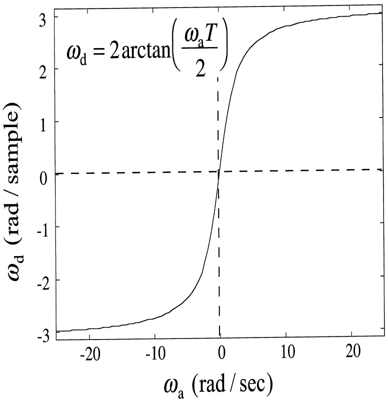
$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$H_d(z) = H_a\left[\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]$$

$$z = \frac{1 + (T/2)s}{1 - (T/2)s}$$

- Mapping between s and z planes
- Mapping between analog and digital frequencies





Bilinear Transformation-Example No. 1

- Design low-pass filter

    1. cutoff frequency $\omega_{dc} = \pi/5$ rad/sample
    2. transition bandwidth and ripple

       $\Delta\omega_d = 0.2\pi$ radians/sample= 0.1 cycles/sample

       $\delta = 0.1$
    3. use Butterworth analog prototype
    4. set $T = 1$
       $|H_d(\omega_d 1)| = 1 - \epsilon$
       $|H_d(\omega_d 2)| = \delta$

- Map digital to analog frequencies

$\omega_a = \frac{2}{T}\tan(\omega_d/2)$

$\omega_{d1} = \pi/5 - 0.1\pi$

$\omega_{a1} = 0.3168$ rad/sec

$\omega_{d2} = \pi/5 + 0.1\pi$

$\omega_{a2} = 1.0191$ rad/sec

- Solve for filter order and cutoff frequency

  $|H_a(\omega_a)|^2 = \frac{1}{1+(\omega_a/\omega_c)^{2N}}$

  $|H_a(\omega_{a1})| = 1 - \epsilon$

  $|H_a(\omega_{a2})| = \delta$

  $$N = \left\lceil \frac{1}{2} \frac{\log[(|H_a(\omega_{a2})|^{-2} - 1)/(|H_a(\omega_{a1})|^{-2} - 1)]}{\log(\omega_{a2}/\omega_{a1})} \right\rceil$$

  $$\omega_{ac} = \frac{\omega_{a2}}{(|H_a(\omega_{a2})|^{-2} - 1)^{1/(2N)}}$$

- Result

  $N = 3$

  $\omega_{ac} = 0.4738$

- Determine transfer function of analog filter

  $H_a(s)H_a(-s) = \frac{1}{1+(s/j\omega_{ac})^{2N}}$

- Poles are given by

  $$s_k = \omega_{ac} e^{j\left(\frac{\pi}{2} + \frac{\pi}{2N} + \frac{2\pi k}{2N}\right)}, \quad k = 0, 1, ..., 2N-1$$

- Take N poles with negative real parts for H(s)

  $s_0 = -0.2369 + j0.4103 = 0.4738e^{j2\pi/3}$

  $s_1 = -0.4738 + j0.0000 = 0.4738e^{j\pi}$

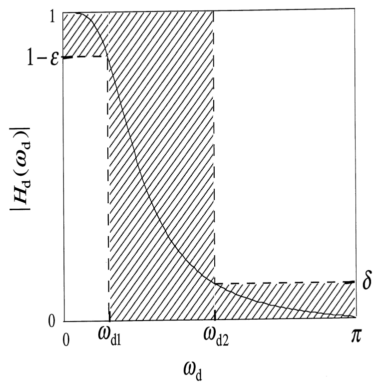
  $s_2 = -0.2369 - j0.4103 = 0.4738e^{-j2\pi/3}$

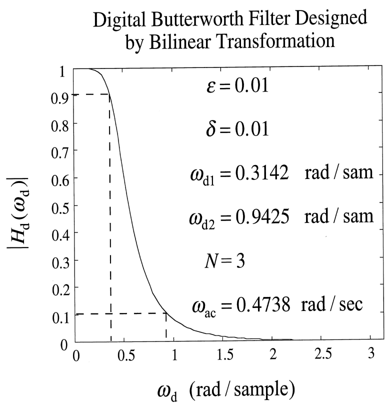
- Transfer function of anlaog filter

  $$H_a(s) = \frac{\omega_{ac}^3}{(s - s_0)(s - s_1)(s - s_2)}$$

- Transform to discrete-time filter
  $H_d(z) = H_a\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$

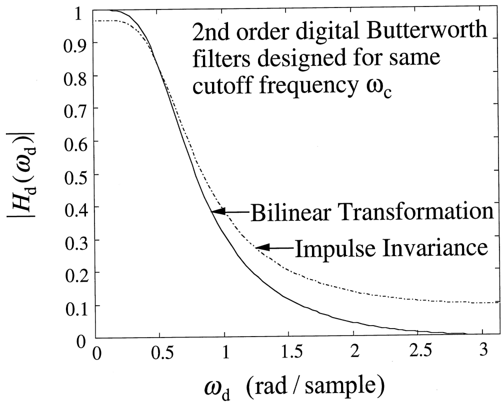
$$H_d(z) = \frac{\omega_{ac}^3(1+z^{-1})^3}{8(1-z^{-1}-s_0)(1-z^{-1}-s_1)(1-z^{-1}-s_2)}$$
$$= \frac{0.0083 + 0.0249z^{-1} + 0.0249z^{-2} + 0.0083z^{-3}}{1.0000 - 2.0769z^{-1} + 1.5343z^{-2} - 0.3909z^{-3}}$$





Bilinear Example No.2

- Design low-pass filter $\omega_{dc} = \pi/5$ rad/sample

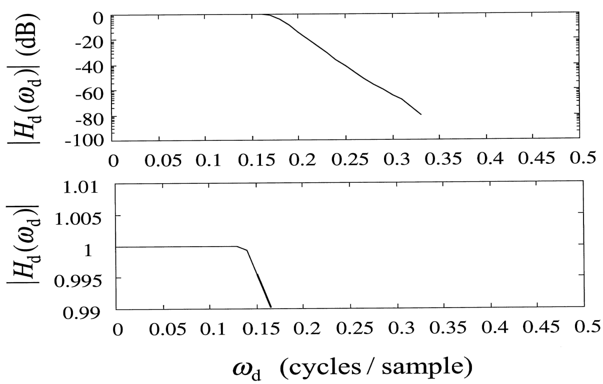


1. Cutoff frequency
2. Transition bandwidth and ripple

   $\Delta\omega_d = 0.1\pi$ radians/sample $= 0.05$ cycles/sample

   $\delta = 0.01$
3. Use Butterworth analog prototype
4. set $T = 1$

- Result:$N = 13$



Frequency Transformations of Lowpass IIR Filters

- Given a prototype lowpass digital filter design with cutoff frequency θ_c, we can transform directly to:
  1. a lowpass digital filter with a different cutoff frequency
  2. a highpass digital filter
  3. a bandpass digital filter
  4. a bandstop digital filter
- General form of transformation $H(z) = H_{lp}(Z)\}_{Z}^{-1} = G(z^{-1})$
- Example - lowpass to lowpass
  1. Transformation
     $Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$
  2. Associated design formula
     $\alpha = \dfrac{\sin\left(\frac{\theta_c - \omega_c}{2}\right)}{\sin\left(\frac{\theta_c + \omega_c}{2}\right)}$

Chapter 8

Random Signals

8.1 Chapter Outline

In this chapter, we will discuss:

1. Single Random Variable
2. Two Random Variables
3. Sequences of Random Variables
4. Estimating Distributions
5. Filtering of Random Sequences
6. Estimating Correlation

8.2 Single Random Variable

Random Signals

Random signals are denoted by upper case X and are completely characterized by density function $f_X(x)$

$P(a \leq X \leq b) = \int_a^b f_X(x)dx$
$\Rightarrow f_X(x)\Delta x = P(x \leq X \leq x + \Delta x)$

The cumulative distribution function is $F_X(x) = P(X \leq x)$

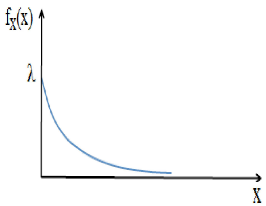
$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$

$\therefore \dfrac{dF_X(x)}{dx} = f_X(x)$

Properties of density function

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Example 1



$f_X(x) = \lambda e^{-\lambda x}u(x)$

$$\begin{aligned}\int_0^{\infty} f_X(x)dx &= \int_0^{\infty} \lambda e^{-\lambda x}dx \\ &= e^{-\lambda x}\Big|_{\infty}^{0} \\ &= 1\end{aligned}$$

What is the probability $X \leq 1$?

$$\begin{aligned}P(X \leq 1) &= \int_0^{1} \lambda e^{-\lambda x}dx \\ &= e^{-\lambda x}\Big|_{1}^{0} \\ &= 1 - e^{-\lambda}\end{aligned}$$

Expectation

$E\{g(X)\} = \int g(x) f_X(x)dx$ where g is a function of X
Special Cases:

1. Mean value
   $g(x) = x$
   $\overline{X} = E\{X\} = \int x f_X(x)dx$

2. Second moment
   $g(x) = x^2$
   $\overline{X^2} = E\{X^2\} = \int x^2 f_X(x)dx$

3. Variance
   $g(x) = (X - \overline{X})^2$

Expectation is a linear operator
$E\{ag(x) + bh(x)\} = aE\{g(x)\} + bE\{h(x)\}$

$$\begin{aligned} \therefore \sigma_X^2 &= E\left\{(X - \overline{X})^2\right\} \\ &= E\left\{X^2 - 2X\overline{X} + \overline{X}^2\right\} \\ &= E\left\{X^2\right\} - \overline{X}^2 \end{aligned}$$

Example 2

$E\{X\} = \int_0^\infty \lambda x e^{-\lambda x} dx$
Integrate by parts
$\int u dv = uv - \int v du$
$u = x \quad du = 1 \quad v = -e^{-\lambda x} \quad dv = \lambda e^{-\lambda x}$

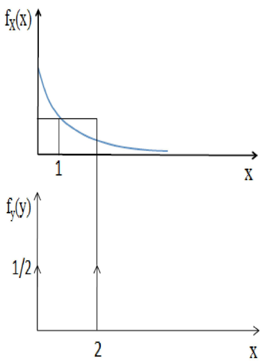
$$\begin{aligned} \int_0^\infty \lambda x e^{-\lambda x} dx &= -x e^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= -x e^{-\lambda x}\Big|_0^\infty - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty \\ &= \frac{1}{\lambda} \end{aligned}$$

Example 3

$E\left\{X^2\right\} = \int_0^\infty \lambda x^2 e^{-\lambda x} dx$
Integrate by parts
$u = x^2 \quad du = 2x \quad v = -e^{-\lambda x} \quad dv = \lambda e^{-\lambda x}$

$$\int_0^\infty x^2 \lambda e^{-\lambda x} dx = -x^2 e^{-\lambda x}\Big|_0^\infty + 2\int_0^\infty x e^{-\lambda x} dx$$
$$= \frac{2}{\lambda}$$

$\sigma_X^2 = \frac{2}{\lambda} - \frac{1}{\lambda^2}$
$\lambda = 1 \quad \overline{X} = 1 \quad \sigma_X^2 = 1 \quad \sigma_X = 1$



X and Y have the same mean and same standard deviation

Transformations of a Random Variable

- $Y = aX + b$

- $\overline{Y} = a\overline{X} + b$

- $$\begin{aligned}\sigma_Y^2 &= E\left\{(Y-\overline{Y})^2\right\}\\ &= E\left\{\left[aX+b-(a\overline{X}+b)\right]^2\right\}\\ &= a^2E\left\{(X-\overline{X})^2\right\}\\ &= a^2\sigma_X^2\end{aligned}$$

- $$\begin{aligned}F_Y(y) &= P\{Y\le y\}\\ &= P\{aX+b\le y\} \quad \text{assume a} > 0\\ &= P\left\{X\le \frac{y-b}{a}\right\}\\ &= F_X\left(\frac{y-b}{a}\right)\end{aligned}$$

- $f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$

In general $f_Y(y) = \left|\frac{1}{a}\right| f_X\left(\frac{y-b}{a}\right)$

8.3 Two Random Variables

1. Joint density function
   $P\{a\le X\le b\cap c\le Y\le d\} = \int_a^b\int_c^d f_{XY}(x,y)dxdy$
2. Joint distribution function
   $F_{XY}(x,y) = \int_{-\infty}^x\int_{-\infty}^y f_{XY}(x,y)dxdy$
3. Marginal densities
   $f_X(x) = \int f_{XY}(x,y)dy$
   $f_Y(y) = \int f_{XY}(x,y)dx$
4. Expectation
   $E\{g(X,Y)\} = \int\int g(x,y)f_{XY}(x,y)dxdy$

5. Linearity
   $E\{ag(x,y)+bh(x,y)\} = aE\{g(x,y)\} + bE\{h(x,y)\}$

6. Correlation
   $g(x,y) = xy$

7. Covariance
   $g(x,y) = (x-\overline{X})(y-\overline{Y}) \quad \overline{XY} = E\{g(x,y)\}$

8. Correlation coefficient

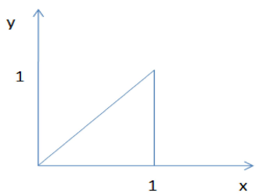
$$\begin{aligned}
\sigma_{XY}^2 &= E\{g(x,y)\} \\
&= \frac{1}{\sigma_X\sigma_Y}\left[E\{XY\} - E\left\{\overline{X}Y\right\} - E\left\{X\overline{Y}\right\} + \overline{X}\,\overline{Y}\right] \\
&= \frac{1}{\sigma_X\sigma_Y}\left\{\overline{XY} - \overline{X}\,\overline{Y}\right\}
\end{aligned}$$

9. Independence

$$\begin{aligned}
f_{XY}(x,y) = f_X(x)f_Y(y)& \\
&\Rightarrow \sigma_{XY}^2 = 0 \text{ and } \overline{XY} = \overline{X}\,\overline{Y}
\end{aligned}$$

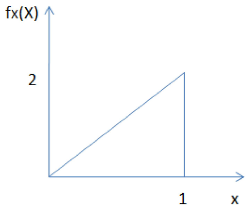
Example 4

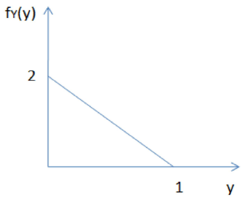
$$f_{XY}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1 \\ 0, & \text{else} \end{cases}$$



$$f_X(x) = \begin{cases} \displaystyle\int_0^x 2dy = 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

$$f_Y(y) = \begin{cases} \displaystyle\int_y^1 2dx = 2(1-y), & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$





$$\begin{aligned}\overline{XY} &= \int_{x=0}^{1}\int_{y=0}^{x} 2xydxdy \\ &= \int_{x=0}^{1} x\left[\int_{y=0}^{x} 2ydy\right]dx \\ &= \int_{x=0}^{1} x^3dx \\ &= \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}\end{aligned}$$

$$\overline{X} = \int_0^1 x(2x)dx = \frac{2}{3}x^3|_0^1 = \frac{2}{3}$$
$$\overline{Y} = \frac{1}{3}$$

$$\begin{aligned}\overline{X^2} &= \int_0^1 x^2(2x)dx \\ &= \left.\frac{2}{4}x^4\right|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}
\therefore \sigma_X^2 &= \frac{1}{2} - \frac{4}{9} \\
&= \frac{1}{18} \\
&= \sigma_y^2
\end{aligned}$$

$$\begin{aligned}
\sigma_{XY}^2 &= \frac{\overline{XY} - \overline{X}\ \overline{Y}}{\sigma_X \sigma_Y} \\
&= \frac{1/4 - 2/9}{1/18} \\
&= \frac{1}{2}
\end{aligned}$$

8.4 Sequences of Random Variables

Characteristics

$X_1, X_2, X_3, ..., X_N$ denote a sequence of random variables

Their behavior is completely characterized by
$P\left\{x_1^l \leq X_1 \leq x_1^u, x_2^l \leq X_2 \leq x_2^u, ..., x_N^l \leq X_N \leq x_N^u\right\}$

$$= \int_{x_1^l}^{x_1^u} \int_{x_2^l}^{x_2^u} ... \int_{x_N^l}^{x_N^u} f_{X_1, X_2, ..., X_N}(x_1, x_2, ..., x_N) dx_1, dx_2, ..., dx_N$$

It is apparent that this can get complicated very quickly
In practice, we consider only 2 cases:

1. $X_1, X_2, ..., X_N$are mutually independent
$$\Rightarrow f_{X_1, X_2, ..., X_N}(x_1, x_2, ..., x_N) = \prod_{i=1}^{N} f_{X_i}(x_i)$$
2. $X_1, X_2, ..., X_N$ are jointly Gaussian

Example 5

$X_1, X_2, ..., X_n$ are i.i.d with mean μ_X and std. deviation σ_X^2
$N_1, N_2, ..., N_n$ are i.i.d with zero mean
$Y_i = aX_i + N_i$

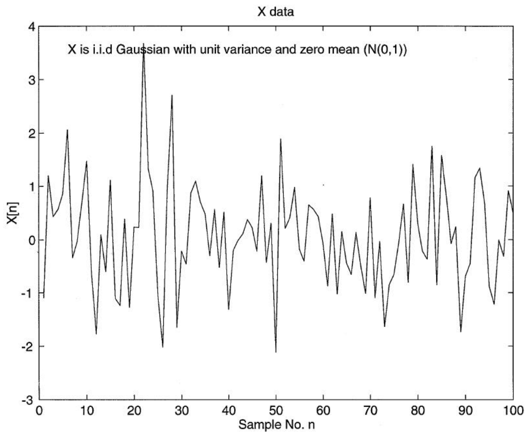
$$E\{Y_i\} = aE\{X_i\} + E\{N_i\}$$
$$\overline{Y} = a\overline{X}$$

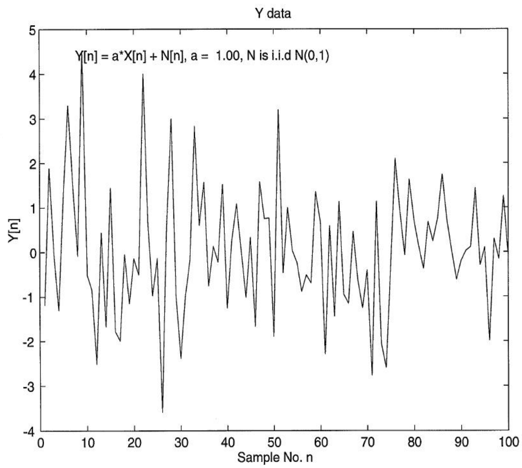
$$\begin{aligned}\overline{Y^2} &= E\left\{(aX_i + N_i)^2\right\} \\ &= a^2\overline{X^2} + \overline{N^2}\end{aligned}$$

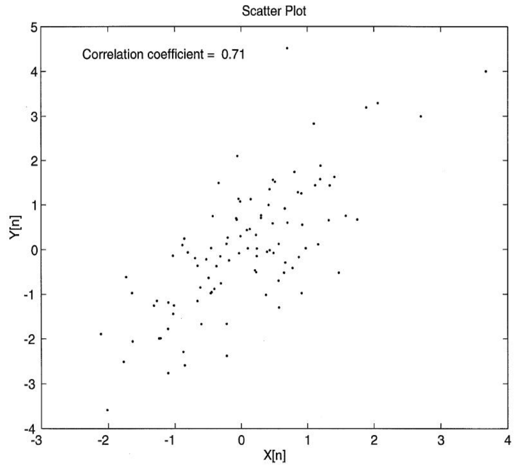
$$\begin{aligned}\sigma_Y^2 &= \overline{Y^2} - \left(\overline{Y}\right)^2 \\ &= a^2\overline{X^2} + \sigma_N^2 - a^2\left(\overline{X}\right)^2 \\ &= a^2\sigma_X^2 + \sigma_N^2\end{aligned}$$

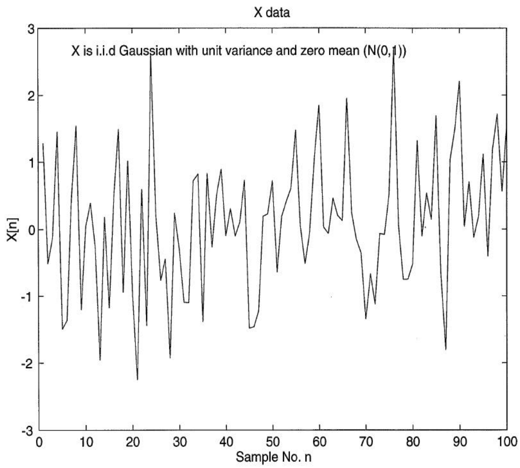
$$\begin{aligned}\overline{XY} &= E\{X_i(aX_i + N_i)\} \\ &= a\overline{X^2}\end{aligned}$$

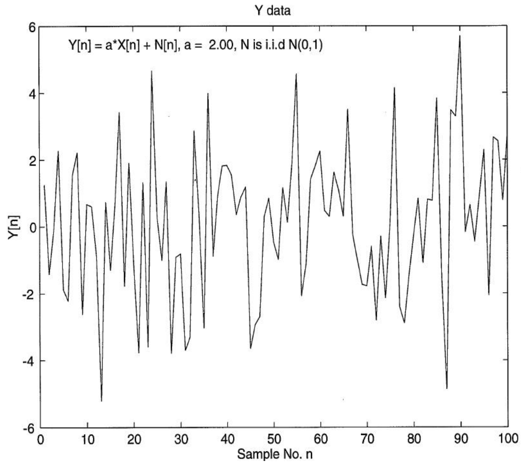
$$\begin{aligned}\sigma_{XY}^2 &= \frac{a\overline{X^2} - a\left(\overline{X}\right)^2}{\sigma_X\left(a^2\sigma_X^2 + \sigma_N^2\right)^{1/2}} \\ &= a\frac{\sigma_X}{\left(\sqrt{a^2\sigma_X^2 + \sigma_N^2}\right)} \\ &= \frac{1}{\sqrt{1 + \frac{\sigma_N^2}{\left(a^2\sigma_X^2\right)}}}\end{aligned}$$

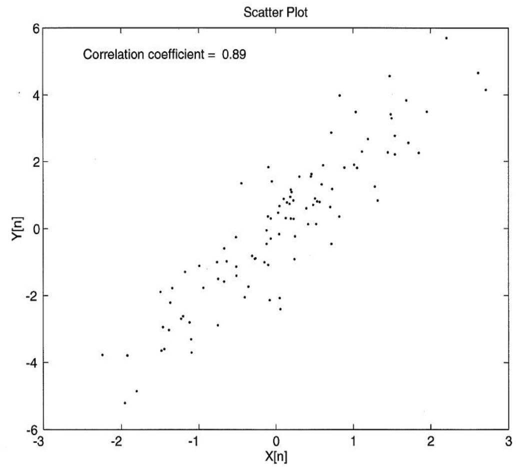

X data
X is i.i.d Gaussian with unit variance and zero mean (N(0,1))
X[n]
Sample No. n
4
3
2
1
0
-1
-2
-3
0 10 20 30 40 50 60 70 80 90 100

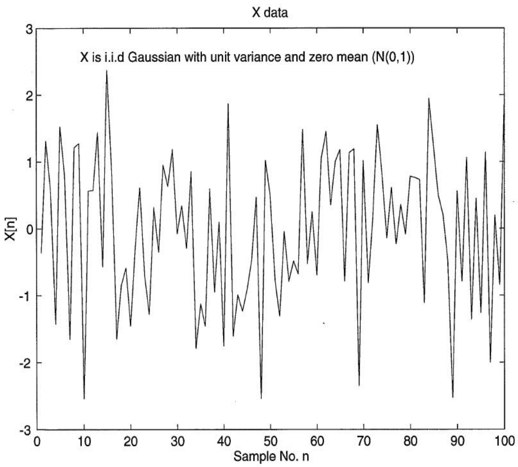

Y data
Y[n] = a*X[n] + N[n], a = 1.00, N is i.i.d N(0,1)
Y[n]
Sample No. n
5
4
3
2
1
0
-1
-2
-3
-4
0 10 20 30 40 50 60 70 80 90 100

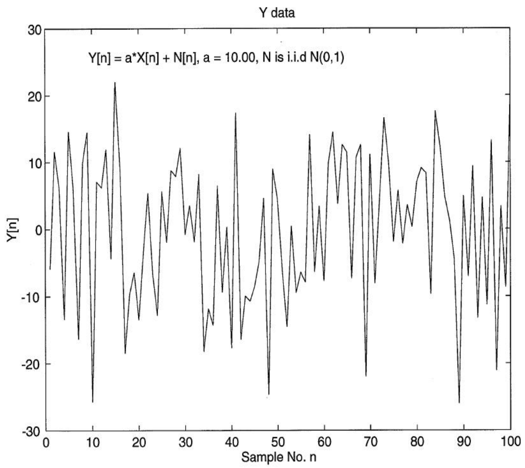

Scatter Plot
Correlation coefficient = 0.71
Y[n]
X[n]
5
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4

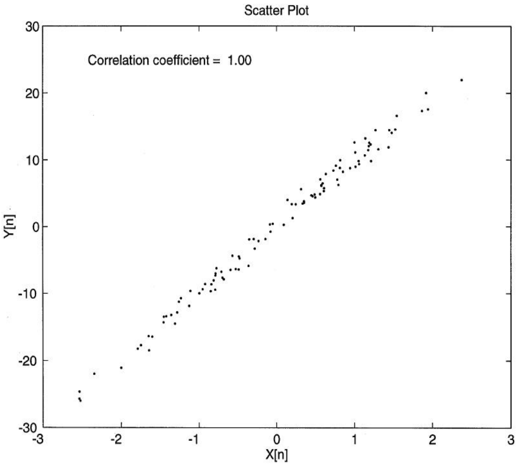

X data
X is i.i.d Gaussian with unit variance and zero mean (N(0,1))
X[n]
Sample No. n
3
2
1
0
-1
-2
-3
0
10
20
30
40
50
60
70
80
90
100


Y data
Y[n] = a*X[n] + N[n], a = 2.00, N is i.i.d N(0,1)
Y[n]
Sample No. n
6
4
2
0
-2
-4
-6
0
10
20
30
40
50
60
70
80
90
100


Scatter Plot
Correlation coefficient = 0.89
Y[n]
X[n]
6
4
2
0
-2
-4
-6
-3
-2
-1
0
1
2
3


X data
X is i.i.d Gaussian with unit variance and zero mean (N(0,1))
X[n]
Sample No. n
3
2
1
0
-1
-2
-3
0
10
20
30
40
50
60
70
80
90
100


Y data
Y[n] = a*X[n] + N[n], a = 10.00, N is i.i.d N(0,1)
Y[n]
Sample No. n
30
20
10
0
-10
-20
-30
0
10
20
30
40
50
60
70
80
90
100



Scatter Plot
Correlation coefficient = 1.00
30
20
10
0
-10
-20
-30
Y[n]
-3
-2
-1
0
1
2
3
X[n]

8.5 Estimating Distributions

So far, we have assumed that we have an analytical model for the probability densities of interest.
This was reasonable for quantization. However, suppose we do not know the underlying densities. Can we estimate it from observations of the signal?

Suppose we observe a sequence of i.i.d rv's

$X_1, X_2, ..., X_N$

let $E(X) = \mu_X$

How do we estimate μ_X?

$\hat{\mu_X} = \frac{1}{N}\sum_{n=1}^{N} X_n$

How good is the estimate?

μ_X is actually a random variable, let's call it Y

$$\begin{aligned} E\{Y\} &= E\left\{\frac{1}{N}\sum_{n=1}^{N} X_n\right\} \\ &= E\{X\} \\ &= \mu_X \end{aligned}$$

$$\begin{aligned} E\left\{|Y-\mu_X|^2\right\} &= E\left\{\left(\frac{1}{N}\sum_{n=1}^{N}|X_n-\mu_X|\right)^2\right\} \\ &= \frac{1}{N^2}\sum_{m=1}^{N} N \sum_{n=1}^{N} E\left\{[X_m-\mu_X][X_n-\mu_X]\right\} \\ &= \frac{1}{N^2}\sum_{m=1}^{N} N\sigma_X^2 \\ &= \frac{1}{N}\sigma_X^2 \end{aligned}$$

How do we estimate variance?

$$\hat{\sigma_X^2} = \frac{1}{N}\sum_{n=1}^{N}[X_n - \mu_X] - [\hat{\mu_X} - \mu_X]]^2$$
$$= \frac{1}{N}\sum_{n=1}^{N}|X_n - \mu_X|^2 - 2[\hat{\mu_X} - \mu_X]\frac{1}{N}\sum_{n=1}^{N}[X_n - \mu_X]$$
$$+ \frac{1}{N}\sum_{n=1}^{N}[\hat{\mu_X} - \mu_X]^2$$
$$= \frac{1}{N}\sum_{n=1}^{N}|X_n - \mu_X|^2 - [\hat{\mu_X} - \mu_X]^2$$

call this r.v Z

$$E\{Z\} = \frac{1}{N}\sum_{n=1}^{N}\sigma_X^2 - E\left\{|\hat{\mu_X} - \mu_X|^2\right\}$$

$$E\left\{|\hat{\mu_X} - \mu_X|^2\right\} = E\left\{\left(\frac{1}{N}\sum_{n=1}^{N}[X_n - \mu_X]\right)^2\right\}$$
$$= \frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=1}^{N}E\{[X_m - \mu_X][X_n - \mu_X]\}$$
$$= \frac{1}{N^2}N\sigma_X^2$$
$$= \frac{1}{N}\sigma_X^2$$

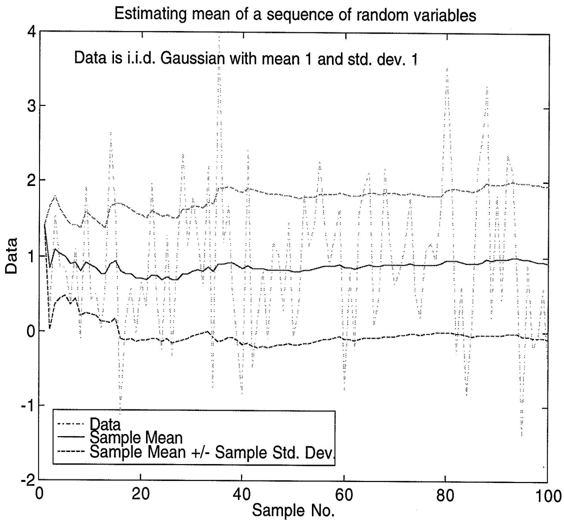
$$\therefore\ E\{Z\} = \sigma_X^2 - \frac{1}{N}\sigma_X^2$$
$$= \sigma_X^2\left(\frac{N-1}{N}\right)$$

So the estimate is not unbiased
We define unbiased estimate as:

$$\hat{\sigma^2_{X_{unb}}} = \frac{N}{N-1}\hat{\sigma_X^2}$$
$$= \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \widehat{\mu_X})^2$$

$$
\begin{aligned}
\sigma_X^2 &= \frac{1}{N}\sum_{n=1}^{N}|X_n - \widehat{\mu_X}|^2 \\
&= \frac{1}{N}\left\{\sum_{n=1}^{N}|X_n|^2 - 2\widehat{\mu_X}\sum_{n=1}^{N}X_n + \widehat{\mu_X}^2\right\} \\
&= \frac{1}{N}\sum_{n=1}^{N}|X_n|^2 - \hat{\mu_X^2}
\end{aligned}
$$

$$
\therefore \sigma_{X_{unb}}^2 = \frac{N}{N-1}\hat{\sigma_X^2}
$$



So far we have seen how to estimate statistics such as mean and variance of a random variable.

These ideas can be extended to estimation of any moments of the random variable. But suppose we want to know the actual density itself.

How do we do this?

We need to compute the histogram.

Consider a sequence of i.i.d r.v's with density $f_X(x)$

We divide the domain of X into intervals

$I_k = \left\{x : \left(k - \frac{1}{2}\right)\Delta \le x \le \left(k + \frac{1}{2}\right)\Delta\right\}$

We then define the r.v.'s

$U^k = \begin{cases} 1, & X \in I_k \\ 0, & \text{else} \end{cases}$

We observe a sequence of i.i.d r.v.'s $X_n \quad n = 1, ..., N$ and compute the corresponding r.v.'s U_n^k

Finally we let

$Z^k = \frac{1}{N} \sum_{n=1}^{N} U_n^k$

Then since this r.v has the same form as the estimator of mean which we analyzed earlier, we know that

$E\left\{Z^k\right\} = E\left\{U_n^k\right\}$

$\sigma_{Z^k}^2 = \dfrac{1}{N}\sigma_{U^k}^2$

Now what is $E\left\{U_n^k\right\}$

$$\begin{aligned} E\left\{U_n^k\right\} &= 1.P\left\{X_n \in I_k\right\} + 0.P\left\{X_n \notin I_k\right\} \\ &= P\left\{X_n \in I_k\right\} \\ &= \int_{\left(k-\frac{1}{2}\right)\Delta}^{\left(k+\frac{1}{2}\right)\Delta} f_X(x)dx \end{aligned}$$

So we see that we are estimating the integral of the density function over the range $\left(k - \frac{1}{2}\right)\Delta \le x \le \left(k + \frac{1}{2}\right)\Delta$

According to the fundamental theorem of calculus, there exists

$\left(k - \frac{1}{2}\right)\Delta \le \tau \le \left(k + \frac{1}{2}\right)\Delta$

such that

$$\begin{aligned} f_X(\tau)\Delta &= \int_{\left(k-\frac{1}{2}\right)\Delta}^{\left(k+\frac{1}{2}\right)\Delta} f_X(x)dx \\ &= E\left\{Z^k\right\} \end{aligned}$$

Assuming $f_X(x)$ is continuous,

As $\Delta \to 0$

$E\left\{Z^k\right\} = f_X(\tau)\Delta \to f_X(k\Delta)\Delta$

defined as

$\phi = \dfrac{\left(\overline{U_k}\right)^2}{\sigma_{U^k}^2} = \overline{U_k}$

For fixed $f_X(x), \overline{U^k} = P\{X \in I_k\}$ will increase with increasing Δ so there is a tradeoff between measurement noise and measurement resolution.

Of course, for fixed Δ and $f_X(x)$, we can always improve our result by increasing N.

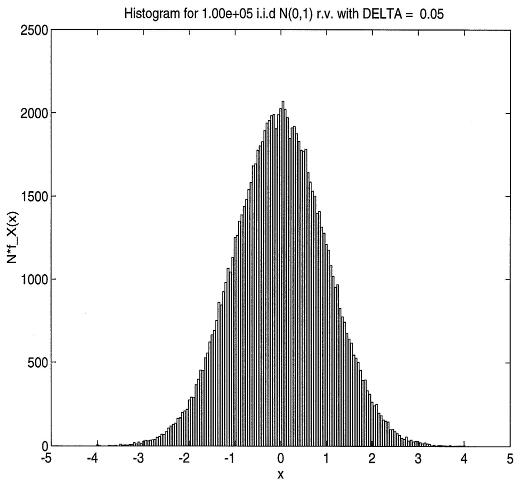
What is the variance of U_n^k?

$$\begin{aligned} E\left\{\left(U_n^k\right)^2\right\} &= 1^2 P\{X \in I_k\} + 0^2 P\{X \notin I_k\} \\ &= P\{X \in I_k\} \\ &= \overline{U_k} \end{aligned}$$

so $\sigma^2_{U^k} = \overline{U_k} - (\overline{U_k})^2 = \overline{U_k}(1 - \overline{U_k})$

We would expect to operate in range where Δ is small so

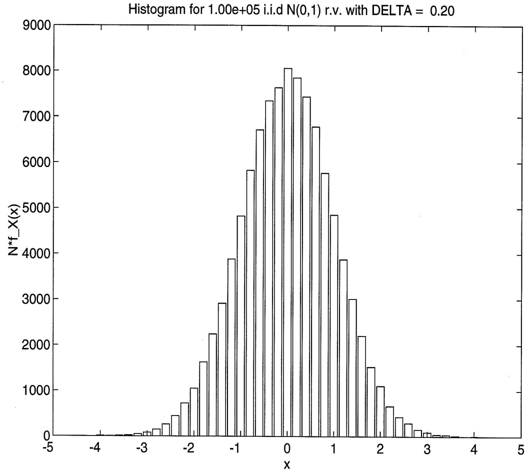
$\sigma^2_{U^k} \approx \overline{U^k}$



8.6 Filtering of Random Sequences

Let us define $\mu_X(n) = E\{X_n\}$

$r_{XX}(m, n) = E\{X_m X_n\}$



Special case

A process is w.s.s. if

1. $\mu_X(n) \equiv \mu_X$
2. $r_{XX}(m,n) = r_{XX}(n-m,0) \equiv r_{XX}(n-m)$

Convention:

$r_{AB}[a,b] = r_{AB}[b-a]$

Consider

$Y[n] = \sum_m h[n-m]X[m]$

$E\{Y_n\} = \sum_m h[n-m]E\{X_m\}$

If X_n is w.s.s

$$\begin{aligned} E\{Y_n\} &= \sum_m h[n-m]\mu_X \\ &= \sum_m h[n]\mu_X \\ &= \mu_X[x] \text{is a constant} \end{aligned}$$

To find r_{YY}, we first define cross-correlation

$$
\begin{aligned}
r_{XY}[m,n] &= E\{X_m, Y_n\} \\
&= E\left\{X_m \sum_k h[n-k]X_k\right\} \\
&= \sum_k h[n-k]E\{X_m X_k\} \\
&= \sum_k h[n-k]r_{XX}[k-m] \\
&\text{let l=k-m } \Rightarrow k = m + l
\end{aligned}
$$

$$
\begin{aligned}
r_{XY}[m,n] &= \sum_k h[n-(m+l)]r_{XX}(l) \\
&= \sum_k h[n-m-l]r_{XX}(l)
\end{aligned}
$$

$\therefore r_{XY}[m,n] = r_{XY}[n-m]$
then

$$
\begin{aligned}
r_{YY}[m,n] &= E\{Y_m Y_n\} \\
&= E\left\{Y_m \sum_k h[n-k]X_k\right\} \\
&= \sum_k h[n-k]r_{XY}[m-k] \\
&\text{let } l = m-k \Rightarrow \;\; k = m-l \\
&= \sum_k h[n-(m-l)]r_{XY}(l) \\
&= \sum_k h[n-m+l]r_{XY}(l) \\
&= \sum_k h[n-m-l]r_{XY}(-l)
\end{aligned}
$$

so $r_{YY}[m,n] = r_{YY}[n-m]$
summarizing:

If X is w.s.s, Y is also w.s.s

Interpretation of Correlation

Recall that for two different random variables X and Y, we defined covariance to be

$\sigma_{XY}^2 = E\left\{(X - \mu_X)(Y - \mu_Y)\right\}$

and correlation coefficient

$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$

as measures of how related X and Y are,i.e whether or not they tend to vary together?

Since $X[m]$ and $X[m+n]$ or $X[m]$ and $Y[m+n]$ are just two different pairs of r.v.'s, we can see that

$r_{XX}[n] = E\left\{X[m]X[m+n]\right\}$

$r_{XY}[n] = E\left\{X[m]Y[m+n]\right\}$

are closely related to covariance and correlation coefficient.

Suppose X and Y have zero mean, these r.v's have the same interpretation. How related are X_m and X_{m+n} or Y_m and Y_{m+n}?

Let's consider some examples:

Suppose X[n] is i.i.d with zero mean.

Since mean of a w.s.s process is just a constant offset or shift, we can let it equal zero without loss of generality.

Then $r_{XX}[n] = E\left\{X[m]X[m+n]\right\}$

$r_{XX}[n] = \begin{cases} \sigma_X^2, & n = 0 \\ 0, & \text{else} \end{cases}$

$r_{XX}[n] = \sigma_X^2 \delta[n]$

Now suppose

$$Y[n] = \sum_{m=0}^{7} 2^{-m} X[n-m]$$

This is a filtering operation with

$h[n] = \{2^{-n}[u[n] - u[n-8]\}$

We already know that

$$\begin{aligned} r_{XY}[n] &= h[n] * r_{XX}[n] \\ &= \sigma_X^2 h[n] \end{aligned}$$

$r_{XY}[n] = E\{X[m]Y[m+n]\}$

What about the sequence $Y[n]$?

Is it i.i.d?

Recall

$$\begin{aligned} r_{YY}[n] &= h[n] * r_{XY}[-n] \\ &= h[n] * \sigma_X^2 h[-n] \\ &= \sigma_X^2 \sum_m h[n-m]h[-m] \\ &= \sigma_X^2 \sum_m h[n+m]h[m] \end{aligned}$$

Cases

1. $n \leq -7$,
   $r_{YY}[n] = 0$

2. $-7 \leq n \leq 0$

$$\begin{aligned} r_{YY}[n] &= \sigma_X^2 \sum_{m=0}^{-n+7} 2^{-(n+m)}2^{-m} = \sigma_X^2 2^{-n} \sum_{m=0}^{-n+7} 2^{-2m} \\ &= \sigma_X^2 2^{-n} \frac{1-2^{-2(-n+8)}}{1-2^{-2}} \\ &= \sigma_X^2 \frac{4}{3}\left[2^{-n} - 2^{(n-16)}\right] \end{aligned}$$

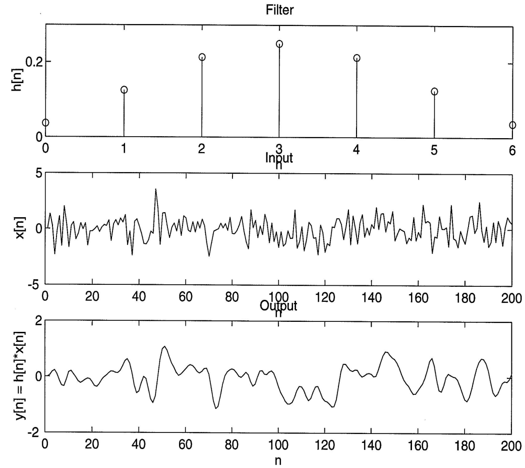
3. $0 \leq n \leq 7$

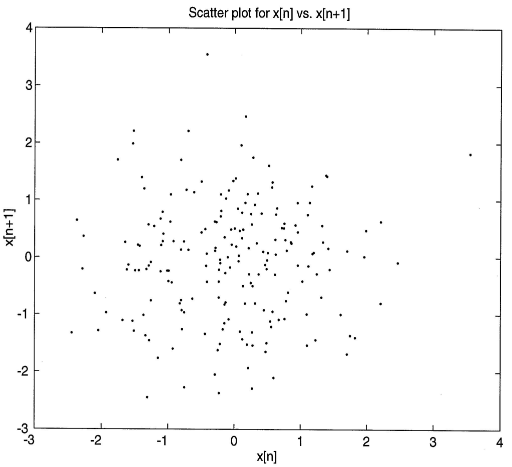
$$\begin{aligned} r_{YY}[n] &= \sigma_X^2 2^{-n} \sum_{m=-n}^{7} 2^{-2m} \quad l = m+n \\ &= \sigma_X^2 2^{-n} \sum_{l=0}^{n+7} 2^{-2(l-n)} \\ &= \sigma_X^2 2^{n} \sum_{l=0}^{n+7} 2^{-2l} = \sigma_X^2 2^n \frac{1-2^{-2(n+8)}}{1-2^{-2}} \\ &= \frac{4}{3}\sigma_X^2[2^n - 2^{-n-16}] \end{aligned}$$

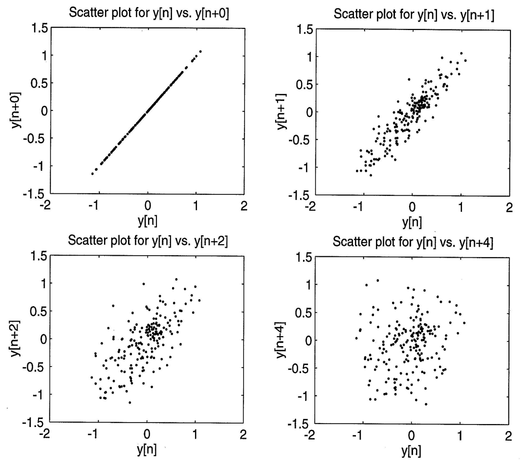
4. $n > 7$
   $r_{YY}[n] = 0$

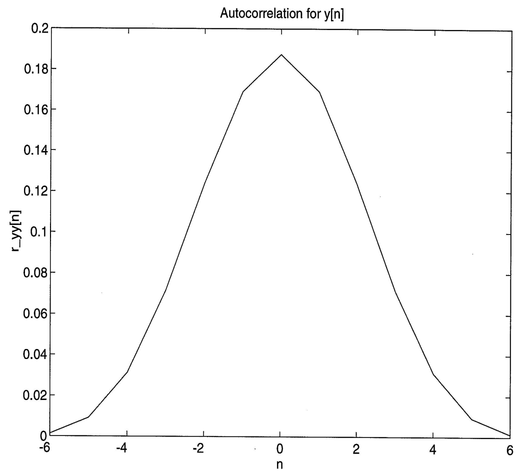
Summarizing

$$r_{YY}[n] = \begin{cases} 0, & n < -7 \\ \frac{4}{3}\sigma_X^2[2^n - 2^{(-n-16)}], & -7 \le n \le 0 \\ \frac{4}{3}\sigma_X^2[2^{-n} - 2^{(n-16)}], & 0 \le n \le 7 \\ 0, & n > 7 \end{cases}$$


Filter
h[n]
0.2
0
0 1 2 3 4 5 6
n
Input
x[n]
5
0
-5
0 20 40 60 80 100 120 140 160 180 200
n
Output
y[n] = h[n]*x[n]
2
0
-2
0 20 40 60 80 100 120 140 160 180 200
n


Scatter plot for x[n] vs. x[n+1]
x[n+1]
4
3
2
1
0
-1
-2
-3
-3 -2 -1 0 1 2 3 4
x[n]


Scatter plot for y[n] vs. y[n+0]
Scatter plot for y[n] vs. y[n+1]
Scatter plot for y[n] vs. y[n+2]
Scatter plot for y[n] vs. y[n+4]
y[n+0]
y[n+1]
y[n+2]
y[n+4]
y[n]


Autocorrelation for y[n]
r_yy[n]
n

8.7 Estimating Correlation

Assume X and Y are jointly w.s.s processes

$E[X_n] = \mu_X$

$E[Y_n] = \mu_X$

$r_{XY}[m, m+n] = E\{X[m]Y[m+n]\}$

$= r_{XY}[n]$

Suppose we form the sum

$$Z[n] = \frac{1}{M}\sum_{m=0}^{M-1} X[m]Y[m+n]$$

Consider

$$E\{Z[n]\} = \frac{1}{M}\sum_{m=0}^{M-1} E\{X[m]Y[m+n]\}$$

$$E\{Z[n]\} = \frac{1}{M}\sum_{m=0}^{M-1} r_{XY}[n] = r_{XY}[n]$$

So we have an unbiased estimator of correlation.

Let's look a little more closely at what the computation involves:

Suppose we computed

$\hat{r_{XY}}[n] = Z[n]$ for the interval $-N \leq n \leq N$

$$= \frac{1}{M}\sum_{m=0}^{M-1} X[m]Y[m+n]$$

For $n = -N$

$X[0]X[1]...X[M-1]$

$Y[-N]Y[-N+1]...Y[-N+M-1]$

For $n = 0$

$X[0]X[1]...X[M-1]$

$Y[0]Y[1]...Y[M-1]$

For $n = N$

$X[0]X[1]...X[M-1]$

$Y[N]Y[N+1]...Y[N+M-1]$

In each case, we used same data set for X, bu the data used for Y varied.

Suppose instead we collect Z sets of data

$X[0]...X[M-1]$

$Y[0]...Y[M-1]$

and use this data to compute estimate for every n

let $X_{tr}[n] = X[n]\{u[n] - u[n-m]\}$

let $Y_{tr}[n] = Y[n]\{u[n] - u[n-m]\}$

Then let

$$\begin{aligned} v[n] &= \sum_m X_{tr}[m]Y_{tr}[m+n] \\ &= \sum_{m=0}^{M-1} X[m]Y[m+n]\{u[m+n] - u[m+n-M]\} \end{aligned}$$

$$\begin{aligned} E\{V[n]\} &= \sum_{m=0}^{M-1} E\{X[m]Y[m+n]\}\{u[m+n] - u[m+n-M]\} \\ &= \sum_{m=0}^{M-1} r_{XY}[n]\{u[m+n] - u[m+n-M]\} \\ &= r_{XY}[n]\sum_{m=0}^{M-1}\{u[m+n] - u[m+n-M]\} \end{aligned}$$

So an unbiased estimate of correlation is given by

$$r_{\hat{X}Y}[n] = \left|\frac{1}{M-n}\right| \sum_{m=0}^{M-1} X[m]Y_{tr}[m+n]$$

∗ Cover image taken from website
http://www.fmwconcepts.com/imagemagick