ECE 438 lecture Wednesday March 8, 2023

Announcements:

- Office Hour today at 3:30p EST
- HW #6 due today via Gradescope at 11:59p EST
- Exam #2 will take place Wednesday 22 March, 2023
  - same format as Exam #1
  - bring photo ID
  - covering HWs 4,5,6 & associated lecture materials

Two conditions for optimality of a nonuniform quantizer

① $y_\ell = E\{X \mid \underline{x_\ell} < X \le \underline{x_{\ell+1}}\} \quad \ell = 0, \ldots, N-1$

② $x_\ell = \dfrac{y_{\ell-1} + y_\ell}{2}$

Comments

1. Can solve these equations iteratively
2. This is called the Lloyd-Max algorithm
3. Can generalize to more than one dimension

For 3 dimensions, algorithm provides an optimal way to quantize RGB images to a small palette of colors (Linde-Buzo-Gray algorithm)

(4) This is a special case of the k-means algorithms, which is a unsupervised clustering method widely used in ML

(5) Matlab (presumably also Numpy) has a built-in function for k-M algorithm

(6) Concept of fixing a set of parameters while optimizing other parameters, and then switch the sets is very powerful

Examples

(1) Expectation-Maximization algorithm

(2) Iterated Conditional Modes (ICM)

(3) Deep Learning (training phase) one example is generator adversarial networks (GAN)
- generator
- discriminator

For examples of L-M algorithm applied to Gaussian densities, see on-line lectures by Prof. Bernd Girod (Stanford U.)

New topic: Two random variables (Module 3, 1, 2 (formatted))

Joint density function $f_{XY}(x,y)$

$$P\{(a < X \leq b) \cap (c < Y \leq d)\} = \int_{x=a}^{b} \int_{y=c}^{d} f_{XY}(x,y)\,dx\,dy$$

properties:

1. $f_{XY}(x,y) \geq 0$
2. $\iint_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

Joint distribution functions

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \underline{f_{XY}(\xi, \eta)\,d\xi\,d\eta}$$

psi, eta

$$= P\{X \leq x, Y \leq y\}$$

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

Marginal densities:

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f_Y(y) = \int_{x=}^{\infty} f_{XY}(x,y)\,dx$$

$$F_X(x) = F_{XY}(x, \infty)$$

$$F_Y(y) = F_{XY}(\infty, y)$$

<u>expectation</u>

$$\mathcal{E}\{g(X,Y)\} = \int\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\,dx\,dy$$

<u>linearity</u>

$$\mathcal{E}\{a\,g(X,Y) + b\,h(X,Y)\} = a\,\mathcal{E}\{g(X,Y)\} + b\,\mathcal{E}\{h(X,Y)\}$$

<u>Special cases</u>

<u>correlation</u>

$$\mathcal{E}\{XY\} = \overline{XY} \quad , \quad g(x,y) = xy$$

$$\therefore \overline{XY} = \int\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy$$

<u>covariance</u>

$$g(x,y) = (x - \bar{X})(y - \bar{Y})$$

$$\sigma^2_{XY} = \mathcal{E}\{(X - \bar{X})(Y - \bar{Y})\}$$

$$= \mathcal{E}\{XY - \bar{X}Y - X\bar{Y} + \bar{X}\bar{Y}\}$$

$$\sigma^2_{XY} = \overline{XY} - \bar{X}\bar{Y}$$

Correlation coefficient

$$\rho_{xy} = \frac{\sigma_{xy}^2}{\sigma_x \sigma_y}$$

It can be shown that $0 \le |\rho_{xy}| \le 1$

Independence of two r.v.s:

def: Two r.v.s. X & Y are statistically independent

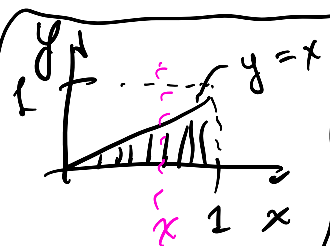
$\Leftrightarrow \quad f_{xy}(x,y) = f_x(x) f_y(y)$

$\Rightarrow \quad \overline{XY} = \bar{X}\bar{Y}$

$\Rightarrow \quad \sigma_{xy}^2 = 0 \Rightarrow \rho_{xy} = 0$

Note $\sigma_{xy}^2 = 0 \not\Rightarrow$ independence of X & Y

example (see posted (legacy) notes)



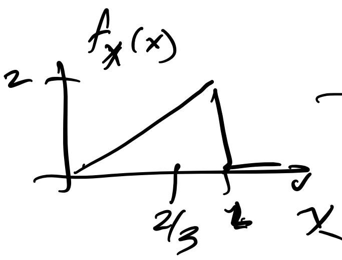
$$f_{xy}(x,y) = \begin{cases} 2, & (x,y) \in \text{[shaded region]} \\ 0, & \text{else} \end{cases}$$

check: $f_{xy}(x,y) \ge 0$

$\iint f_{xy}(x,y)\, dx\, dy = 1$

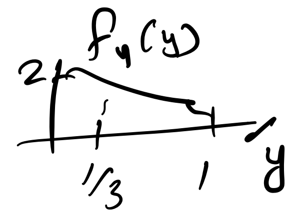
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_0^x 2\,dy & 0 \le x \le 1 \\ 0, & x \le 0 \text{ or } x \ge 1 \end{cases}$$

$$= \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$



without showing details:

$$f_Y(y) = \begin{cases} (2-2y) & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$



$$\overline{XY} = \iint xy\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1 x \left\{ \int_0^x y\, 2\,dy \right\} dx$$

$$= \int_0^1 x\{x^2\}\,dx = \int_0^1 x^3\,dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}$$

$$\overline{X} = \int_0^1 x(2x)\,dx = \int_0^1 2x^2\,dx = \left.\frac{2}{3}x^3\right|_0^1 = \frac{2}{3}$$

by symmetry $\overline{Y} = 1/3$

$$\overline{X^2} = \int_0^1 x^2\, 2x\,dx = \int_0^1 2x^3\,dx = \left.\frac{x^4}{2}\right|_0^1 = \frac{1}{2}$$

$\sigma_X^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

by symmetry $\sigma_Y^2 = \frac{1}{18}$ also

$\sigma_{XY}^2 = \overline{XY} - \bar{X}\bar{Y} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$

$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}^2}{\sigma_X^2} = \frac{1/36}{1/18} = \frac{1}{2}$

end of Module 3.1.2

end of 2 r.v.'s

next: Module 3.1.1.2 - sequences of r.v.'s
(handwritten)

A sequence of r.v.'s is a random signal