ECE 438 lecture   Monday 6 March 2023

Announcements

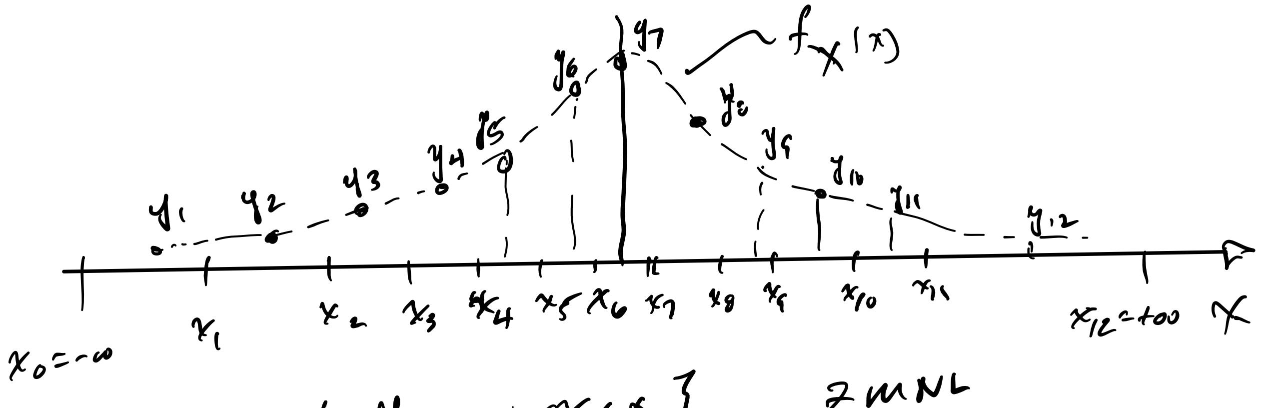
(1) Office Hours today:

- 2:30p EST
- 4:00p EST

(2) Quiz 3 - to be released today at 6p EST
due on Gradescope at 11:59p EST today.
You have 30 minutes to complete it after
downloading it from Gradescope

(3) HW #6 - due on Gradescope by March 8 at
11:59p EST

Nonuniform Quantization



$Q(x) = \{ y_k , \; x_{k-1} \leq x \leq x_k \}$ ZMNL

Need to determine values for the parameters
$x_0, \ldots x_N$ $N = 12$ in this example

$y_1, \ldots, y_N$

Module 3.1.1.2 ⇐ formally prepared

Criterion:

minimize $\overset{\text{phi}}{\phi} = \mathcal{E}\{|x - q(x)|^2\}$

$$\boxed{= \sum_{k=0}^{N} \int_{x_k}^{x_{k+1}} [x - y_k]^2 f_X(x)\,dx}$$

To solve problem, differentiate with respect to unknown parameters (one-by one) & set derivatives $= 0$ (This is only a necessary condition for an optimal solution.)

lets pick a $k = l$ (l - fixed)

differentiate ϕ with respect to y_l:

$$\frac{\partial \phi}{\partial y_l} = \frac{\partial}{\partial y_l}\left\{ \int_{x_l}^{x_{l+1}} [x - y_l]^2 f_X(x)\,dx \right\}$$

$$= \int_{x_l}^{x_{l+1}} \frac{\partial}{\partial y_l}\left\{ [x - y_l]^2 f_X(x) \right\} dx$$

$$= \int_{x_l}^{x_{l+1}} 2[x - y_l] \overbrace{\frac{\partial}{\partial y_l}\{[x - y_l]}^{-1} f_X(x)\,dx = 0$$

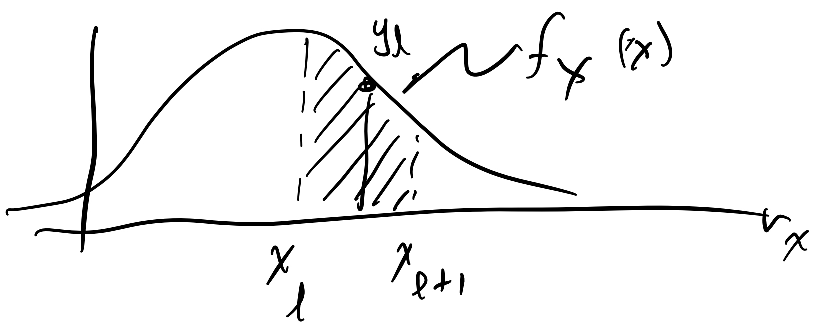
set $\frac{\partial \phi}{\partial y_l} = 0$

$$\int_{x_\ell}^{x_{\ell+1}} x f_X(x)\,dx = y_\ell \int_{x_\ell}^{x_{\ell+1}} f_X(x)\,dx$$

$$y_\ell = \frac{\int_{x_\ell}^{x_{\ell+1}} x f_X(x)\,dx}{\int_{x_\ell}^{x_{\ell+1}} f_X(x)\,dx}$$

$$f_{X \mid x_\ell \le X \le x_{\ell+1}}(x \mid x_\ell \le X \le x_{\ell+1})$$

$$= E\{X \mid x_\ell \le X \le x_{\ell+1}\}$$



Now find optimal thresholds x_k, $k = 1, \dots, N$

fix $k = \ell$

Consider $\frac{\partial \phi}{\partial x_\ell}$

Review:

let $G(x) = \int_{-\infty}^{x} g(\xi)\,d\xi$ psi indefinite integral

$$\Rightarrow \frac{d\,G(x)}{dx} = g(x)$$

let

$$\beta(x; y_\ell) = \int_{-\infty}^{x} [\zeta - y_\ell]^2 f_X(\zeta)\, d\zeta$$

now $\phi \overset{\text{phi}}{=} \sum_{k=0}^{N-1} \left[\beta(x_{k+1}; y_k) - \beta(x_k; y_k) \right]$

$$\frac{\partial \phi}{\partial x_\ell} = \frac{\partial}{\partial x_\ell} \{ \underbrace{\beta(x_\ell; y_{\ell-1})}_{k=\ell-1} \} - \frac{\partial}{\partial x_\ell} \{ \underbrace{\beta(x_\ell; y_\ell)}_{k=\ell} \}$$

terms $k = \ell-1$, $k = \ell$ involve x_ℓ

what is derivative?:

$$\frac{\partial}{\partial x_\ell} \{ \beta(x_\ell; y_{\ell-1}) \} = [x - y_{\ell-1}]^2 f_X(x)$$

$$- [(x - y_\ell]^2 f_X(x)$$

set it $= 0$

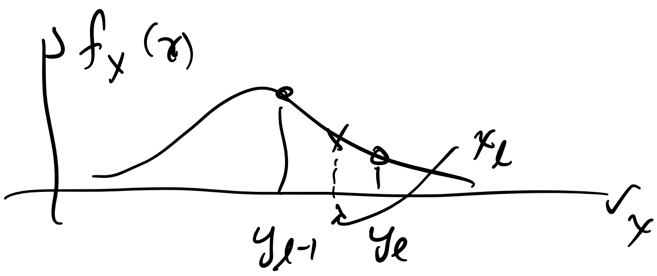
so we have

$$(x - y_{\ell-1})^2 = (x - y_\ell)^2$$

2 solutions:

1. $x - y_{\ell-1} = x - y_\ell$
2. $(x - y_{\ell-1}) = -(x - y_\ell)$

2nd solution $\Rightarrow$ $x_\ell = \dfrac{y_{\ell-1} + y_\ell}{2}$



Have 2 coupled sets of equations:

(1) $y_\ell = \mathcal{E}\{X \mid x_\ell \le X < x_{\ell+1}\}$

$\ell = 0, \ldots, N$

(2) $x_\ell = \dfrac{y_{\ell-1} + y_\ell}{2}$, $\ell = 1, \ldots$

How to solve?

Iterated conditional modes (ICM)

Lloyd-Max algorithm

- first published in 1960 by Max
- Later it was discovered that Lloyd has proposed the idea in an unpublished memorandum
- There is a built-in Matlab function for this

L-M algorithm:

(1) Initialization: Choose N+1 levels y_ℓ to be uniformly spaced with interval Δ

(2) update thresholds $x_\ell = \frac{y_{\ell-1} + y_\ell}{2}$

(3) update output levels $y_\ell = \mathcal{E}\{X \mid x_\ell \le X \le x_{\ell+1}\}$

(4) Check for completion:

(a) Did anything change very much?

(b) Did we complete the specified number of iterations?

If "yes", stop

If "no", repeat steps 2, 3, 4

Comments

(1) If we know $f_X(x)$, can solve the problem analytically.

(2) If we don't know $f_X(x)$, can estimate it by observing many instances of the r.v. X & generate a histogram.

(3) L-M algorithm can be generalized to higher dimensions:

For example, if we have 3 r.v.s R, G, & B

we can optimally quantize a color image to a small number of output levels
(Linde-Buzo-Gray) algorithm