# ECE 438 Lecture    3 March 2023

## Announcements

- HW #6 is now posted
  It is due on Wednesday 8 March at 11:59p EST
  via gradescope
- Solution to HW No. 5 should be posted soon

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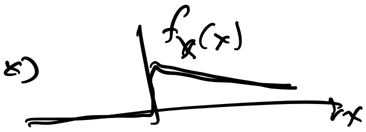
One random variable (r.v.)

$$P\{a \le X \le b\} = \int_a^b f_X(x)\,dx$$

Example: $f_X(x) = \lambda e^{-\lambda x} u(x)$




Does $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$?

use integration by parts:

$$\int u\,dv = uv - \int v\,du$$

$u = x$, $du = 1$

$dv = \lambda e^{-\lambda x}$   $v = -e^{-\lambda x}$

$$\therefore \int \lambda x e^{-\lambda x}\,dx = \underbrace{-x e^{-\lambda x}\Big|_0^\infty}_{=0} + \underbrace{\int_0^\infty e^{-\lambda x}\,dx}_{-\frac{1}{\lambda} e^{-\lambda x}\big|_0^\infty}$$

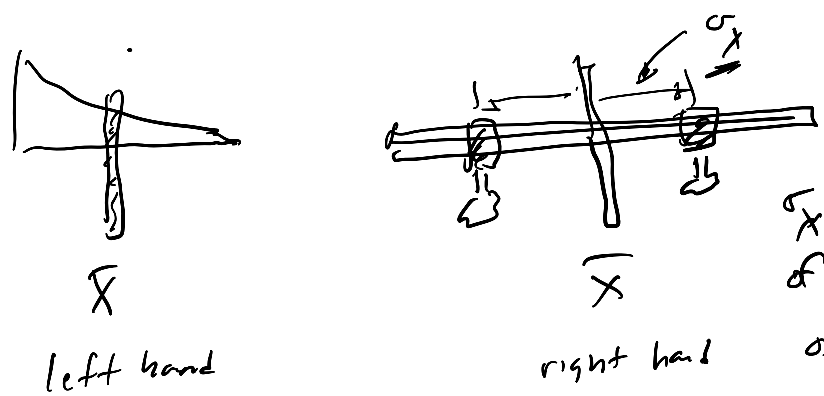
$$= \frac{1}{\lambda} = \bar{X}$$

recall $\bar{X} = E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx$

Similarly (see on-line notes)

$$\overline{X^2} = E\{X^2\} = \frac{2}{\lambda}$$

$$\therefore \sigma_X^2 = \overline{X^2} - (\overline{X})^2 = \frac{2}{\lambda} - \left(\frac{1}{\lambda}\right)^2$$




$\sigma_X$ - is a measure of the moment of inertia

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Module 3.1.1.1

Transformation of r.v.'s (linear)

Given a r.v. $X$, define a second r.v. $Y$ according to

$Y = aX + b$ , where $a$ & $b$ are constants

$$\overline{Y} = E\{Y\} = E\{aX + b\}$$

$= a E\{X\} + b$ by linearity of expectation

$= a\overline{X} + b$

$$\sigma_Y^2 = E\{(Y - \overline{Y})^2\}$$

$$= E\{[aX + b - (a\overline{X} + b)]^2\}$$

$$= a^2 E\{(X-\bar{X})^2\} = a^2\sigma_X^2$$

Lastly, what about $f_Y(y)$? How is this related to $f_X(x)$?

Consider $F_Y(y) = P\{Y \le y\} = P\{\underline{aX+b \le y}\}$

Assume $\underline{\underline{a > 0}}$ $\quad aX+b \le y \iff X \le \frac{y-b}{a}$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{d}{dy}\left\{F_X\left(\frac{y-b}{a}\right)\right\}$$

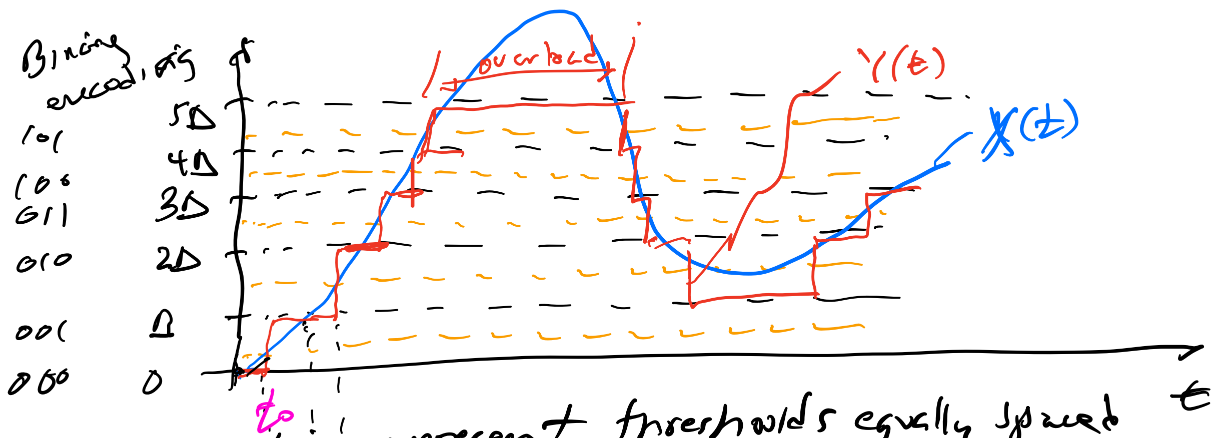
$$= f_X\left(\frac{y-b}{a}\right)\frac{d}{dy}\left\{\frac{y-b}{a}\right\}$$ from chain rule

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

In general $f_Y(y) = \left|\frac{1}{a}\right| f_X\left(\frac{y-b}{a}\right)$ a can be positive or negative

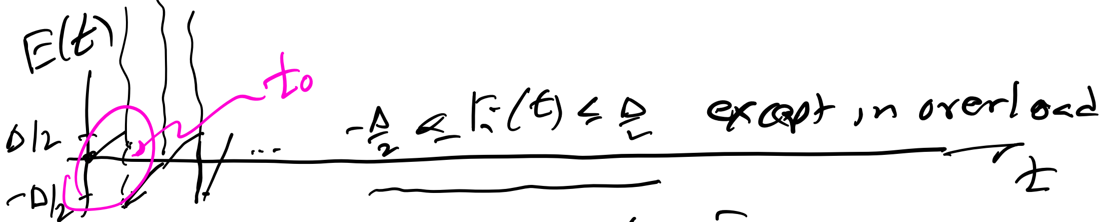
## Quantization

Consider $X(t)$ a continuous-time signal and see what happens when we quantize it




Orange lines represent thresholds equally spaced between adjacent levels

define quantization error $E(t) = Y(t) - X(t)$
$= Q(X(t)) - X(t)$




$-\frac{\Delta}{2} \le E(t) \le \frac{\Delta}{2}$ except in overload

What is $\bar{E}$? looks like $\bar{E} = 0$

How big is $E(t)$?

$$e_{ms} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |E(t)|^2 \, dt$$

Consider the interval from $t = 0$ to $t = t_0$:

$$e_{ms} = \frac{1}{t_0} \int_0^{t_0} \left[ \frac{\Delta}{2t_0} t \right]^2 dt$$

$$= \frac{1}{t_0} \Big|_0^{t_0} \frac{\Delta^2}{4t_0^2} \frac{t^3}{3} = \frac{\Delta^2}{12 t_0^3} \Big|_0^{t_0} t^3 = \frac{\Delta^2}{12}$$

Note that this does not depend on $t_0$!

So to extent that linear approximation for error waveform is valid, $e_{ms} = \frac{\Delta^2}{12}$

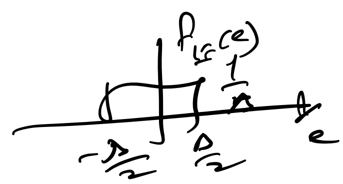
$\Rightarrow$ Have a sufficient number of output levels that are closely enough the $X(t)$ appears nearly linear between output levels

Another viewpoint!

What is a reasonable density function $f_E(e)$?

By inspection of $E(t)$, it spends an equal amount of time at every point between $-\frac{\Delta}{2}$, & $\frac{\Delta}{2}$

$\Rightarrow f_E(e) = \frac{1}{\Delta}\,\mathrm{rect}\left(\frac{e}{\Delta}\right)$




What are statistics for $E$?

$\overline{E} = 0$

$$e_{ms} = 2\int_0^{\Delta/2} \left(\frac{1}{\Delta}\right) e^2\, de = \frac{2}{\Delta}\,\frac{(\Delta/2)^3}{3} = \frac{\Delta^2}{12}$$

$\frac{e^3}{3}$

Same answer as before

Calculate SNR:

$$SNR = \frac{\overline{X^2}}{e_{ms}}$$

What is $\overline{X^2}$?

Assume that $X(t)$ is uniformly distributed over its range

$$-\frac{N-1}{2}\Delta, \text{ to } \frac{N-1}{2}\Delta$$

Note we are assuming that $N$ is odd to make quantizer symmetric.

$$\overline{X^2} = \frac{N^2\Delta^2}{12}$$

$$\therefore \quad SNR = \frac{\overline{X^2}}{e_{rms}} = \frac{N^2\Delta^2/12}{\Delta^2/12} = N^2$$

Further, we assume that $N = 2^B \approx N-1$

$B$ no. bits for quantizer

for large $N$:

$$\text{then } SNR = 2^{2B}$$

now in dB:

$$SNR_{dB} = 10 \log_{10}\left(2^{2B}\right) = 10 \cdot 2 \cdot B \cdot \underbrace{\log_{10}(2)}_{0.3} = 6B \text{ dB}$$