ECE 438 Lecture – Friday 10 March 2023

Announcements:

- No class, labs, or office hours next week
- Exam #2 will be given on Wednesday 20 March 2023

Module 3.1.1.2 (handwritten)

Sequences of r.v's

A sequence of r.v's is a DT random signal!

Behavior is completely characterized by the joint density function:

$$P\{x_1^{(l)} < X_1 \le x_1^{(u)},\ x_2^{(l)} < X_2 \le x_2^{(u)}, \ldots, x_N^{(l)} < X_N \le x_N^{(u)}\} =$$

$$\int_{x_1^{(l)}}^{x_1^{(u)}} \int_{x_2^{(l)}}^{x_2^{(u)}} \cdots \int_{x_N^{(l)}}^{x_N^{(u)}} f_{X_1 X_2 \ldots X_N}(x_1, x_2, \ldots, x_N)\, dx_1\, dx_2 \ldots dx_N \quad (1)$$

This can get messy very quickly

Two special cases:

① $X_1, X_2, \ldots, X_N$ are independent r.v.'s

$\Rightarrow \underline{f_{X_1, X_2, \ldots, X_N}(x_1, x_2, \ldots, x_N)} = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_N}(x_N)$

This is a consequence of independence. It does $\Rightarrow$ independence.

② $X_1, X_2, \ldots, X_N$ are jointly Gaussian
we will <u>not</u> consider this in ECE 438

<u>Example</u>:

$X_1, \ldots, X_n$ are independent and identically distributed (iid)

↑ note switch

with mean $\bar{X} = \mu_X$ and variance σ_X^2

$N_1, \ldots, N_n$ are also i.i.d with mean zero and variance σ_N^2

Assume X_j & N_i are mutually independent for all $i, j \in \{1, \ldots, n\}$

Observe $Y_i = a X_i + N_i$, where $i = 1, \ldots, n$

↑ gain

<u>Question</u>: what can we determine about X_i from looking at Y_i?

$$\bar{Y} = \mathcal{E}\{Y_i\} = \mathcal{E}\{aX_i + N_i\} = a\underbrace{\mathcal{E}\{X_i\}}_{\mu_x} + \cancel{\mathcal{E}\{N_i\}}^{0}$$

$$= a\mu_x = a\bar{X}$$

$$\overline{XY} = \mathcal{E}\{X_i\cdot(aX_i + N_i)\} = a\{X_i^2\} + \mathcal{E}\{\underline{X_i\cdot N_i}\}$$

Note that Y_i's are $\underline{\text{i.i.d.}}$

$$= a\overline{X^2} + \bar{X}\cancel{\bar{N}}^{0} = \underline{a\overline{X^2}}$$

$$\underline{\sigma_{XY}^2} = \mathcal{E}\{(X_i - \mu_x)(Y_i - \bar{Y})\}$$

$$= \overline{XY} - \bar{X}\bar{Y} = a\overline{X^2} - \bar{X}a\bar{X} = \underline{a\sigma_x^2}$$

note that $\mu_x = \bar{X}$

$$\underline{\overline{Y^2}} = \mathcal{E}\{(aX_i + N_i)^2\}$$

$$= \mathcal{E}\{(aX_i)^2 + 2aX_iN_i + N_i^2\}$$

$$= a^2\overline{X^2} + 2a\bar{X}\cancel{\bar{N}}^{0} + \sigma_N^2$$

$$= a^2\overline{X^2} + \sigma_N^2$$

$$\sigma_Y^2 = \overline{Y^2} - (\bar{Y})^2 = a^2\overline{X^2} + \sigma_N^2 - a^2(\bar{X})^2$$

$$= a^2\sigma_X^2 + \sigma_N^2$$

$$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$$

$$= \frac{a\sigma_X^2}{\sigma_X\sqrt{a^2\sigma_X^2 + \sigma_N^2}} \quad \frac{1/(a\sigma_X^2)}{1/(a\sigma_X^2)}$$

$$= \frac{1}{\sqrt{1 + \frac{\sigma_N^2}{a^2\sigma_X^2}}}$$

<u>define</u> $\mathrm{SNR} = \frac{a^2\sigma_X^2}{\sigma_N^2}$

$$\Rightarrow \boxed{\rho_{XY} = \frac{1}{\sqrt{1 + 1/\mathrm{SNR}}}}$$

as $\mathrm{SNR}\uparrow$, $\rho_{XY} \nearrow 1$

as $\mathrm{SNR}\downarrow$, $\rho_{XY} \searrow 0$

Filtering of random sequences

Module 3.1.5 (handwritten)

define $X = \mathcal{E}\{X_n\}$

n : independent variable

new: define $r_{XX}[m,n] = \mathcal{E}\{X_n X_m\}$ autocorrelation function

Special case:

A process (sequence of r.v.'s) is wide-sense stationary (w.s.s.) iff

1. $\overline{X}_n = \overline{X}$ (mean is constant)
2. $r_{XX}[m,n] = r_{XX}[n-m, 0]$

intuitively this says that underlying behavior of process is independent of ~~true~~ origin

Convention: $r_{AB}[a,b] = r_{AB}[b-a]$

Consider

$$Y_n = \sum_{m=-\infty}^{\infty} h[n-m] X_m$$

$$\mathcal{E}\{Y_n\} = \mathcal{E}\left\{\sum_{m=-\infty}^{\infty} h[n-m] X_m\right\}$$

$$= \sum_{m=-\infty}^{\infty} h[n-m]\, \mathcal{E}\{X_m\}$$

If $\bar{X}_m = \bar{X}$, then $\mathcal{E}\{Y_n\} = \bar{Y} = \bar{X} \sum_{m=-\infty}^{\infty} h[n-m]$

or $\bar{Y} = \bar{X} \sum_{m=-\infty}^{\infty} h[m]$

Also, want to know $r_{YY}[m,n]$

start with $\mathcal{E}\{X_m Y_n\} = r_{XY}[m,n]$

$$r_{XY}[m,n] = \mathcal{E}\left\{X_m \sum_{k=-\infty}^{\infty} h[n-k]\, X_k\right\}$$

$$= \sum_{k=-\infty}^{\infty} h[\underline{n-k}]\, \underline{\mathcal{E}\{X_m X_k\}}$$