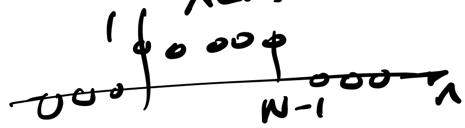
ECE 438 Lecture 1/27/2023

DTFT

$$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

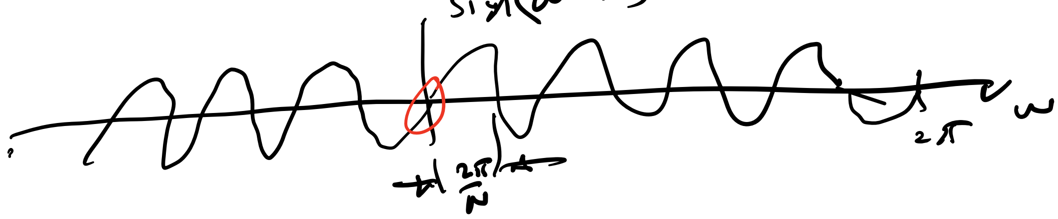
Example

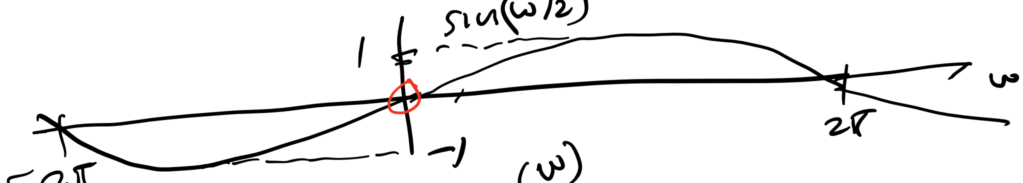
$$x[n]=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{else}\end{cases}$$

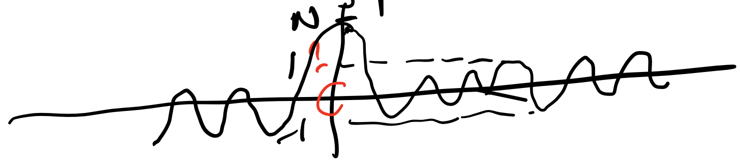


$$X(\omega) = \text{psinc}_N(\omega)\, e^{-j\omega(N-1)/2}$$

$$\text{psinc}_N(\omega)=\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$



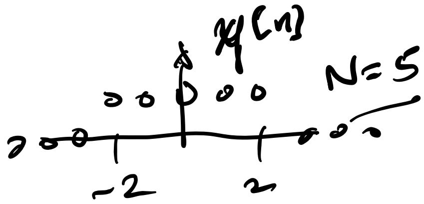




Taylor series expansion for $\sin(\theta) = \theta + \ldots$

$$\lim_{\omega \to 0} \text{psinc}_N(\omega) = \lim_{\omega \to 0} \frac{\omega N/2}{\omega/2} = N$$

for N odd, $x[n]$ has an odd number of points $=1$ $\Rightarrow$ $\frac{N-1}{2}$ is an integer



$$y[n] = x\left[n + \frac{N-1}{2}\right]$$

$$\Rightarrow Y(\omega) = X(\omega) e^{j\left(\frac{N-1}{2}\right)\omega}$$

$$\therefore Y(\omega) = \text{psinc}_N(\omega)$$

psinc function is defined on formulas sheet

This material is from Module 1.3.3.

<u>NEW TOPIC</u> (Module 1.4)

Continuous-time signals & systems (CT)

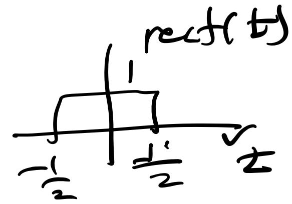
Why study CT in a course on DSP?

<u>Ans 1:</u> we live in an analog or continuous-parameter world.

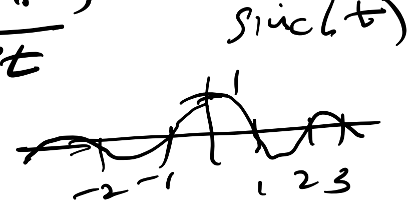
<u>Ans. 2:</u> The theory of CT signals & systems is more symmetric than that of DT signals & systems

Special CT signals:

① $\mathrm{rect}(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & \text{else} \end{cases}$

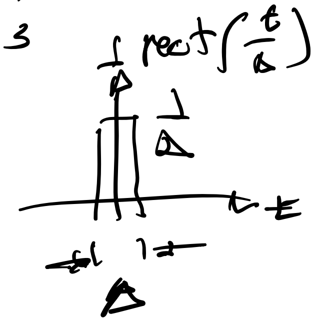


② $\mathrm{sinc}(t) \triangleq \dfrac{\sin(\pi t)}{\pi t}$



why not $\dfrac{\sin(t)}{t}$?

③ $\delta(t) = \lim\limits_{\Delta \to 0} \dfrac{1}{\Delta}\,\mathrm{rect}\left(\dfrac{t}{\Delta}\right)$



area $A_\delta = 1$ for all Δ

properties of $\delta(t)$

ⓐ $\int_{-\infty}^{\infty} x(t)\,\delta(t - t_0)\,dt = x(t_0)$

sifting or sampling property

ⓑ $x(t) * \delta(t - t_0) = x(t - t_0)$

i.e. $\int x(\tau)\,\delta(\tau - (t - t_0))\,d\tau = x(t - t_0)$

ⓒ $\delta(at + b) \equiv \dfrac{1}{|a|}\,\delta(t + b/a)$

proof:

$$\int_{-\infty}^{\infty} x(t)\,\delta(at+b)\,dt =$$

let $\tau = at \Rightarrow d\tau = a\,dt \Rightarrow dt = \frac{d\tau}{a}$

(assume $a>0$) $t = \tau/a$ $\quad t = \frac{\tau}{a}$

have $\frac{1}{a}\int_{-\infty}^{\infty} x\left(\frac{\tau}{a}\right)\delta(\tau+b)\,d\tau = x\left(\frac{b}{a}\right)\frac{1}{a}$

note that also by sifting property

$$\int_{-\infty}^{\infty} x(t)\,\frac{1}{|a|}\delta(t+b/a)\,\cancel{dt} = \frac{1}{|a|}\,x\left(\frac{b}{a}\right)$$

accounts for $a<0$

so $\delta(at+b) \equiv \frac{1}{|a|}\delta(t+b/a)$

Continue with CTFT:

Forward: $X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt$

Inverse: $x(t) = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df$

note symmetry

why not $\omega = 2\pi f$ $\frac{\text{rad}}{\text{sec}}$? f - Hz $\frac{\text{cycles}}{\text{sec}}$

1. more symmetric

② Hz is more intuitive than $\frac{\text{radians}}{\text{sample}}$

Properties

① linearity

$$a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{CTFT} a_1 X_1(f) + a_2 X_2(f)$$

② scaling & shifting

$$x\left(\frac{t - t_0}{a}\right) \xleftrightarrow{CTFT} a X(af) e^{-j2\pi f t_0}$$

no "a"

③ modulation

$$e^{j2\pi f_0 t} x(t) \xleftrightarrow{CTFT} X(f - f_0)$$

④ Initial values

ⓐ $$X(0) = \int_{-\infty}^{\infty} x(t) \underbrace{e^{-j2\pi f t}}_{1} dt \Big|_{f=0}$$

$$= \int_{-\infty}^{\infty} x(t) dt = A_x$$

ⓑ $$x(0) = \int_{-\infty}^{\infty} X(f) \underbrace{e^{j2\pi f t}}_{\equiv 1} df \Big|_{t=0}$$

$$= \int_{-\infty}^{\infty} X(f) df = A_X$$

(5) Parseval

$$\int |x(t)|^2 dt = \int |X(f)|^2 df$$

or $E_x = E_X$

(6) Reciprocity

Suppose $x(t) \xleftrightarrow{CTFT} X(f)$

Note: let $y(t) = X(t)$

then $Y(f) = x(-f)$

(7) Convolution

$$\int x_1(\tau)\, x_2(t-\tau)\, d\tau \xleftrightarrow{CTFT} X_1(f)\, X_2(f)$$

$$x_1(t) * x_2(t) \xleftrightarrow{CTFT} X_1(f)\, X_2(f)$$

(8) Product

$$x_1(t)\, x_2(t) \xleftrightarrow{CTFT} X_1(f) * X_2(f)$$

Transform pairs

(1) $\delta(t) \xleftrightarrow{CTFT} 1$

$$X(f) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt = 1$$

(2) $1 \leftrightarrow \delta(f)$ by reciprocity

③ $e^{j2\pi f_0 t} \cdot \underline{1} \leftrightarrow \delta(f - f_0)$

④ $\cos(2\pi f_0 t) = \frac{1}{2}\{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\}$

$$\cos(2\pi f_0 t) \xleftrightarrow{CTFT} \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$$

note that CTFT is generally not periodic in f $\Rightarrow$ don't need to restrict domain of f

⑤ $\mathrm{rect}(t) \xleftrightarrow{CTFT} \mathrm{sinc}(f)$

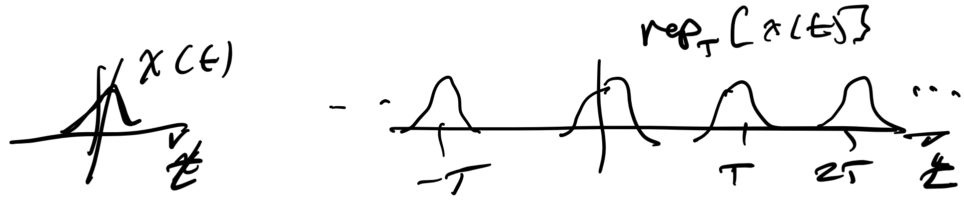
⑥ $\mathrm{sinc}(t) \xleftrightarrow{CTFT} \mathrm{rect}(f)$

by reciprocity

Now we introduce two special operators:

① replication

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$



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