ECE 438 Lecture

1/25/2023

Announcements

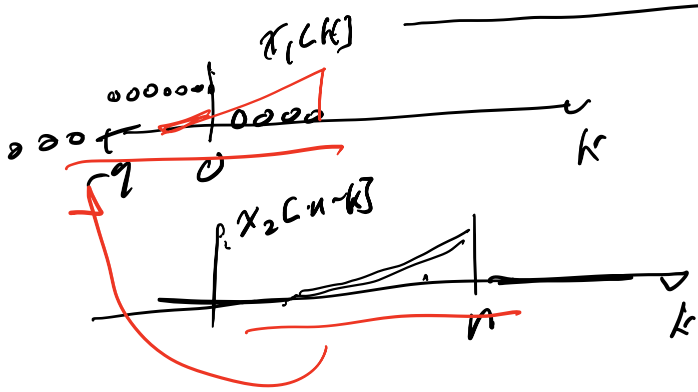
- Office Hours today at 12:30p via Zoom
- HW #1 due tonight at 11:59p via Gradescope

Continuing with Module 2.2

$$x_1[n] = u[n - -9] - u[n]$$

$$x_2[n] = 2^{-n} u[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, \underbrace{x_2[n-k]}_{x_2[-(k-n)]}$$



Cases:

1. $n < -9$
2. $-9 \le n \le 0$
3. $0 < n$

Step 5 Combine cases

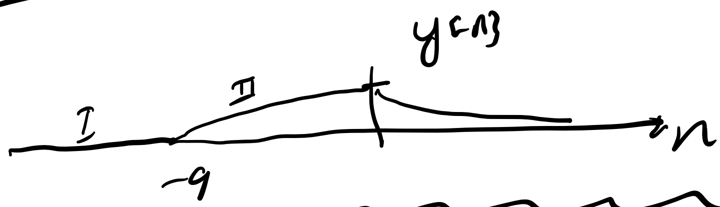
$$y[n] = \begin{cases} 0, & n < -9 \\ 2[1 - 2^{-(n+10)}] & -9 \le n \le 0 \quad \text{II} \\ 1023 \cdot 2^{-(n+9)}, & 0 < n \quad \text{III} \end{cases}$$

Check:

$$y[0] = 2\left(\frac{1023}{1024}\right) \text{ from Case II}$$

$$y[0] = 2\left(\frac{1023}{1024}\right) \text{ from Case III}$$

Step 6 Sketch result



Discrete-Time Fourier Transform (DTFT)

Module 1.3.3

- argument in legacy notes is expressed as $X(e^{j\omega})$
- Here, we write it as $X(\omega)$

When solving HW problems, use only concepts that have been covered in the lecture prior to the due date of the HW.

DTFT - "bread & butter" transforms for ECE 438

Sufficient conditions for existence of DTFT:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{or} \quad \sum_{n=-\infty}^{\infty} [x[n]]^2 < \infty$$

Forward $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ ω - $\frac{\text{rad.}}{\text{sample}}$

continuous-parameter

Inverse $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

DT

Note that $X(\omega) = X(\omega + 2\pi k)$ for integer k

Properties

① linearity

$a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{DTFT} a_1 X_1(\omega) + a_2 X_2(\omega)$

note double-headed arrow

② shifting

$x[n-n_0] \xleftrightarrow{DTFT} X(\omega) e^{-j\omega n_0}$

③ modulation

$e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(\omega - \omega_0)$

④ Parseval's relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

⑤ Initial value

(a) $x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)d\omega$

(b) $X(0) = \sum_{n=-\infty}^{\infty} x[n]$ $\Longleftarrow$ important for image processing

(6) Convolution

$$\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] \xleftrightarrow{DTFT} X_1(\omega)\, X_2(\omega)$$

(7) Product

$$x_1[n]\, x_2[n] \xrightarrow{DTFT} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mu)\, X(\omega-\mu)d\mu$$

Consider again convolution

let $y[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$

find $Y(\omega) = \sum_n y[n] e^{-j\omega n}$

$$= \sum_n \left(\sum_k x_1[k]\, x_2[n-k]\right) e^{-j\omega n}$$

interchange order of summations

$$Y(\omega) = \sum_k x_1[k] \left\{\underbrace{\sum_n x_2[n-k] e^{-j\omega n}}\right\}$$

$e^{-j\omega k} X_2(\omega)$

$= \left\{ \sum_k x_1[k] e^{-j\omega k} \right\} X_2(\omega)$

$Y(\omega) = X_1(\omega) X_2(\omega)$ QED

Application to LTI systems

Special transform pairs: $X(\omega) = \sum_k x[k] e^{-j\omega k}$

① $\delta[n] \xrightarrow{DTFT} 1$

② $1 \xrightarrow{DTFT} ?$ Consider $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

Let $X(\omega) = \delta(\omega)$

then $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega$

$= \frac{1}{2\pi}(1)$

$1 \xrightarrow{DTFT} 2\pi \delta(\omega)$ $\quad -\pi \leq \omega \leq \pi$

or $1 \xrightarrow{DTFT} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ $\quad -\infty < \omega < \infty$

③ $e^{j\omega_0 n} \xrightarrow{DTFT} 2\pi \delta(\omega - \omega_0)$, $-\pi \leq \omega \leq \pi$

by modulation property

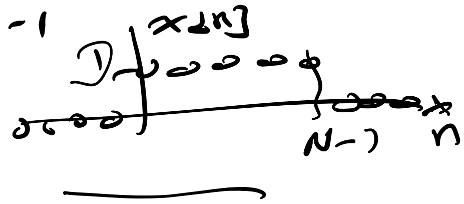
(4) $\cos(\omega_0 n) = \frac{1}{2}\left\{ e^{j\omega_0 n} + e^{-j\omega_0 n} \right\}$

$$\cos(\omega_0 n) \longleftrightarrow \pi\left\{ \delta[\omega-\omega_0] + \delta[\omega+\omega_0] \right\}$$

$$-\pi \le \omega < \pi$$

DTFT of $\sin(\omega_0 n)$ can be determined in a similar way

(5) $x[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{else} \end{cases}$



$$X(\omega) = \sum_{n=0}^{N-1} 1\, e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{\frac{1}{j2}\left(e^{j\omega N/2} - e^{-j\omega N/2}\right) e^{-j\omega N/2}}{\frac{1}{j2}\left(e^{+j\omega/2} - e^{-j\omega/2}\right) e^{-j\omega/2}}$$

$$= \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2}$$

What does this look like?

Module 1.3.3

define $\text{psinc}_N(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$

periodic (Prof. Bouman)

$\sin(\omega N/2) = 0$, whenever $\frac{\omega N}{2} = k\pi$ k - integer

$\Rightarrow \omega = \frac{2\pi k}{N}$

$\sin(\omega/2) = 0$, whenever $\frac{\omega}{2} = k\pi \Rightarrow \omega = 2\pi k$