ECE 438 Lecture

1/18/2023

Systems

Linearity

Time - Invariance

Example 3

$$y[n] = \begin{cases} x[n], & n \geq 0 \\ 0, & \text{else} \end{cases}$$

introduce notation:

$$y[n] = x[n]_+$$

To prove linearity,

hypothesize that

$$x_1[n] \rightarrow y_1[n] = x_1[n]_+$$

$$x_2[n] \rightarrow y_2[n] = x_2[n]_+$$

Form $x_3[n] = \underbrace{a_1 \cancel{x_1}[n] + a_2 x_2[n]}$

Find $y_3[n] = x_3[n]_+$

$$= a_1 x_1[n]_+ + a_2 x_2[n]_+$$

$$= a_1 y_1[n] + a_2 y_2[n]$$

$\therefore$ system is Linear!

For TI consider

$$y[n] = \begin{cases} x[n], & n \geq 0 \\ 0, & \text{else} \end{cases}$$

Hypothesize that system is not TI

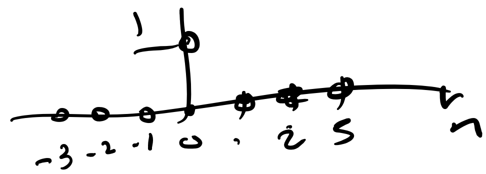
$\Rightarrow$ need to find counter example

Let $x_1[n] = \delta[n]$

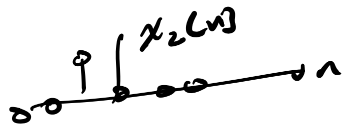
definition unit sample or DT

impulse

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases}$$



$$x_2[n] = \delta[n+1] = \delta[n-(-1)]$$



$$y_2[0] = 0 \neq y_1[n-(-1)]$$

So system is not TI

Why do we care about linearity and TI?

Ans: for linear, TI systems, have the following

characterization:

$$Y(\omega) = H(\omega)\, X(\omega)$$

frequency response

DTFT of output DTFT of input

Consider a new signal

$$x[n] = e^{-j\omega_0 n}$$ complex exponential

ω_0 = frequency units radians/sample

trig identities:

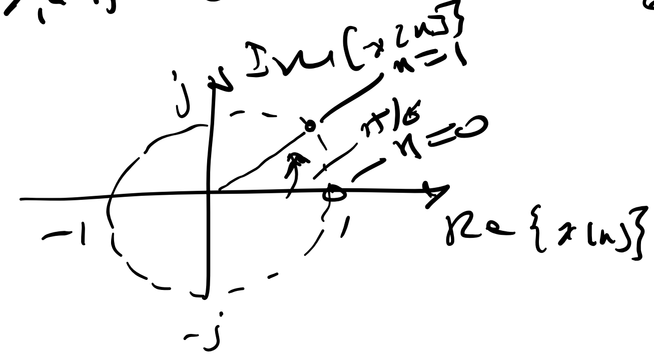
$$e^{j\omega_0 n} = \cos(\omega_0 n) + j \sin(\omega_0 n)$$

imaginary unit

$$\cos(\omega_0 n) = \frac{1}{2}\left\{e^{j\omega_0 n} + e^{-j\omega_0 n}\right\}$$

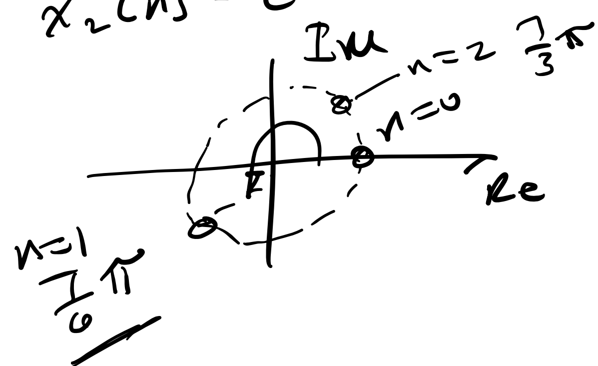
$$\sin(\omega_0 n) = \frac{1}{j2}\left\{e^{j\omega_0 n} - e^{-j\omega_0 n}\right\}$$

Example 1 $\quad x_1[n] = e^{j\frac{\pi}{6}n} \qquad \omega_0 = \frac{\pi}{6}\ \frac{\text{rad.}}{\text{sample}}$



Example 2

$$x_2[n] = e^{j\frac{7\pi}{6}n}$$



Example 3 $\quad \omega_0 = -\frac{5\pi}{6}$

$n=1 \quad \angle = -\frac{5}{6}\pi$

$x_{-2}[1] \equiv x_{13}[1] \Rightarrow$ This is a problem

It is called aliasing

$\Rightarrow$ limit range of frequencies of interest to be

$0 \leq \omega \leq 2\pi$

or $-\pi \leq \omega \leq \pi$

also $e^{j\omega_0 n} = e^{j(\omega_0 n + 2\pi k)}$

for any integer k

Properties of complex numbers:

$z = z_R + j z_I$ Cartesian form

$z = A e^{j\theta}$ Polar coordinate form

def $z = z^* = z_R - j z_I$ complex conjugate

$= A e^{-j\theta}$

properties

$A = \sqrt{z z^*}$

$$z_R = \frac{1}{2}\{z + z^*\}$$

$$z_I = \frac{1}{j2}\{z - z^*\}$$

$$\theta = \arctan\left(\frac{z_I}{z_R}\right)$$ radians

Consider again the 2-pt. MA

$$y[n] = \frac{1}{2}\{x[n] + x[n-1]\}$$

showed that this system is LTI

Consider $x[n] = e^{jwn}$

then $y[n] = \frac{1}{2}\{e^{jwn} + e^{jw(n-1)}\}$

$$= \frac{1}{2}\{e^{jwn} + e^{jwn}e^{-jw}\}$$

$$= \underbrace{\frac{1}{2}\{1 + e^{jw}\}}\, \underline{\underline{e^{jwn}}}$$

so $y[n] = H(w)\, x[n]$

only true for a LTI systems with a complex exponential input!

$$H(w) = \frac{1}{2}\{1 + e^{-jw}\}$$

what is $|H(w)|^2$?

$$|H(\omega)| = \sqrt{|H_{Re}(\omega)|^2 + |H_{Im}(\omega)|^2}$$

<u>not</u> the right approach

again $H(\omega) = \frac{1}{2}\{1 + e^{-j\omega}\}$

need to use a trick :

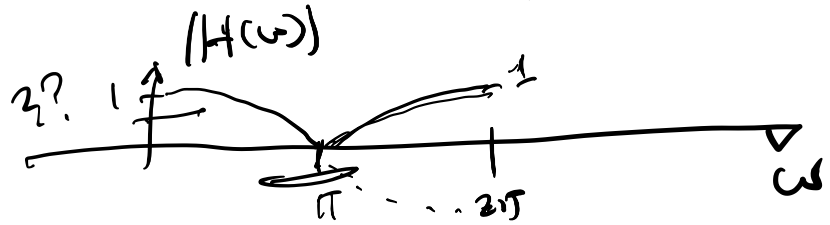
factor out $\frac{1}{2}$ angle

$$H(\omega) = \frac{1}{2}\{e^{j\omega/2} + e^{-j\omega/2}\}e^{-j\omega/2}$$

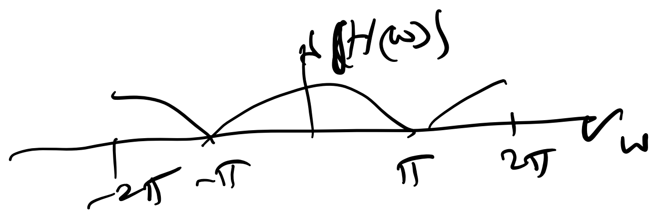
$$= \cos\left(\frac{\omega}{2}\right)e^{-j\omega/2}$$

so $$|H(\omega)| = \left|\cos\left(\frac{\omega}{2}\right)\right| \left|e^{-j\omega/2}\right|$$

$$= \left|\cos\left(\frac{\omega}{2}\right)\right|$$



This is a low-pass filter (LPF)



Fact: for any LTI systems, if the

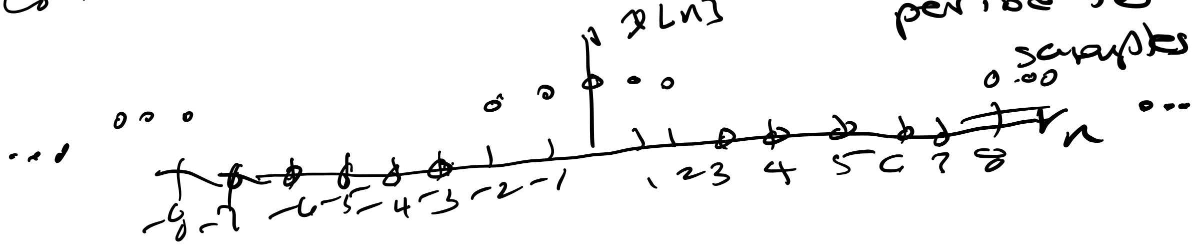
response to a real-valued input is real-valued, then we have the following properties:

① $|H(\omega)| = |H(-\omega)|$ even

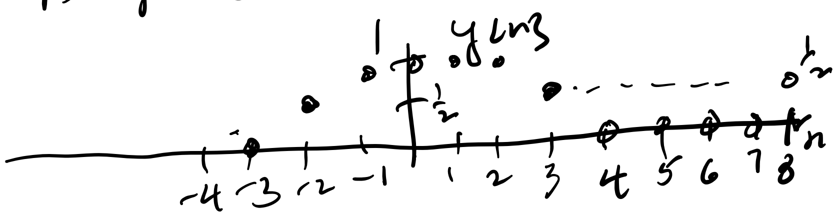
② $\angle H(\omega) = -\angle H(-\omega)$ odd

$$y[n] = \frac{1}{2}\{x[n] + x[n-1]\}$$

Consider the following input $x[n]$. Square wave with period 10 samples



What is $y[n]$? For a 2pt. MA



What has system done?

① smoothed input

② $\frac{1}{2}$ sample delay