ECE 438 lecture Feb. 8, 2023

Announcements

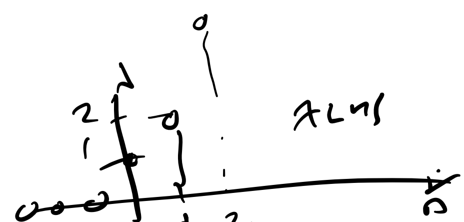
- Office Hour today at 3:30p EST
- HW #3 due today at 11:59p EST via Gradescope
- Notes from 2/6/2023 lecture are posted

ZT – Module 1.5

def $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ 2-sided ZT

Example 1

$x[n] = 2^n u[n]$



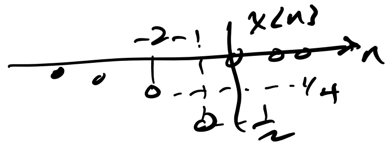
$X(z) = \frac{1}{1 - 2z^{-1}}, \quad |z| > 2$

Example 2

$x[n] = -2^n u[-n-1]$

$= -2^n u[-(n - -1)]$

flip shift to -1



$X(z) = \sum_{n=-\infty}^{-1} x[n] z^{-n}$ let $m = -n$

$$= -\sum_{m=+\infty}^{1} 2^{-m} z^{+m}$$

$$= -\sum_{m=1}^{\infty} 2^{-m} z^{m} = -\sum_{m=1}^{\infty} \left(\frac{2}{z}\right)^{-m}$$

$$= -\sum_{m=0}^{\infty} \left(\frac{2}{z}\right)^{-m} + 1$$

$$= \frac{-1}{1 - z/2} + 1, \quad \left|\frac{z}{2}\right| < 1$$

$$X(z) = \frac{1}{1-2z^{-1}}, \quad |z| < 2$$

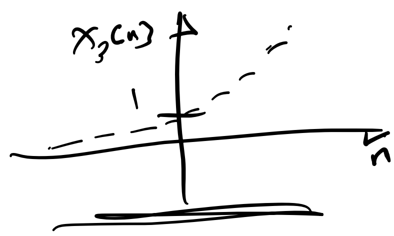
Compare

✓ $x_1[n] = 2^n u[n]$, $X_1(z) = \frac{1}{1-2z^{-1}}, \ |z| > 2$

✓ $x_2[n] = -2^n u[-n-1]$, $X_2(z) = \frac{1}{1-2z^{-1}}, \ |z| < 2$

Example 3

$x_3[n] = 2^n$



$x_3[n] = x_1[n] - x_2[n]$

$\therefore\ X_3(z) = X_1(z) - X_2(z) \equiv 0$ by linearity of ZT also $X_1(z)$ & $X_2(z)$ have no common region of convergence!

$\therefore$ there is no ZT for $x_3[n]$

Thus, we cannot work with 2-sided ZT unless we consider region of convergence

Module 1.5.3 ZT properties and pairs

1. linearity

$$a_1 x_1[n] + a_2 x_2[n] \xrightarrow{ZT} a_1 X_1(z) + a_2 X_2(z)$$

2. shifting

$$x[n-n_0] \xrightarrow{ZT} z^{-n_0} X(z)$$

3. modulation

$$z_0^n x[n] \xrightarrow{ZT} X(z/z_0)$$

4. multiplication by time index

$$n x[n] \xrightarrow{ZT} -z \frac{d}{dz}\{X(z)\}$$

5. $x[n] * y[n] \xrightarrow{ZT} X(z) Y(z)$

6. Relation to DTFT

If $X_{ZT}(z)$ exists for $|z|=1$, then

$$X_{DTFT}(\omega) = X_{ZT}(z)\Big|_{z=e^{j\omega}}$$

note in legacy notes, we write

$$X_{DTFT}(e^{j\omega}) = X_{ZT}(e^{j\omega})$$

Important transform pairs

① $\delta[n] \xleftrightarrow{ZT} 1$

② $a^n u[n] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}}, \quad |z| > |a|$

"a" could be complex-valued. See Ex 1

③ $-a^n u[-n-1] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}}, \quad |z| < |a|$

(see Example 2)

④ $n a^n u[n] \xleftrightarrow{ZT} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$

⑤ $-n a^n u[-n-1] \xleftrightarrow{ZT} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| < |a|$

New topic: Module 1.5.4

ZT and linear, constant coefficient difference equations

Skip theory and go to examples:

System $y[n] = x[n] - x[n-1] - \frac{1}{2} y[n-2]$

$$Y(z) = X(z) - z^{-1} X(z) - \frac{1}{2} z^{-2} Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 + \frac{1}{2} z^{-2}}$$ transfer function

get rid of negative powers of z and factor denominator

$$H(z) = \frac{z(z-1)}{(z^2+\frac{1}{2})^2} = \frac{z(z-1)}{(z - j/\sqrt{2})(z + j/\sqrt{2})}$$

roots of numerator polynomial are called zeros

zeros

roots of denominator polynomial are called poles

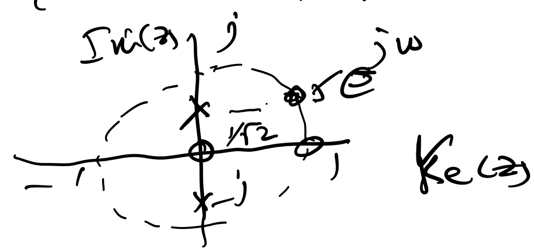
Relation between poles & zeros & frequency response for LTI systems described by linear, constant coefficient difference equations

$$h[n] \xrightarrow{DTFT} H_{DTFT}(\omega) = H_{ZT}(z)\Big|_{z=e^{j\omega}}$$

Return to previous example

$$y[n] = x[n] - x[n-1] - \tfrac{1}{2} y[n-2]$$

$$H(z) = \frac{z(z-1)}{(z - j/\sqrt{2})(z + j/\sqrt{2})}$$



$$H_{zT}(e^{j\omega}) = \frac{e^{j\omega}(e^{j\omega}-1)}{(e^{j\omega} - j/\sqrt{2})(e^{j\omega} + j/\sqrt{2})}$$

$$|H_{zT}(e^{j\omega})| = \frac{|e^{j\omega}|\,|e^{j\omega} - 1|}{|e^{j\omega} - j/\sqrt{2}|\,|e^{j\omega} + j/\sqrt{2}|}$$

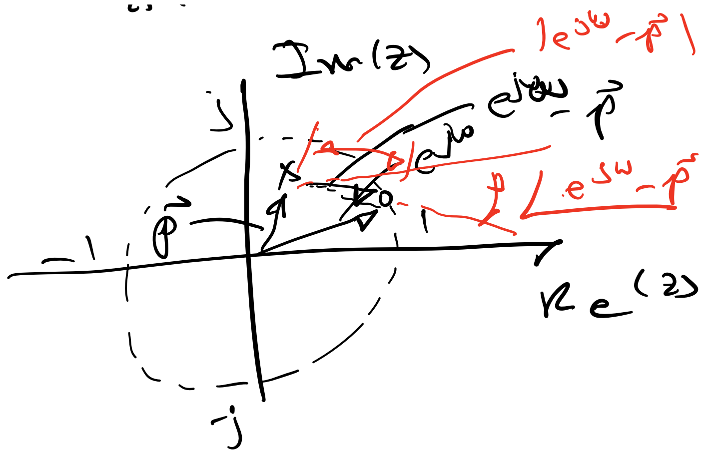
$$\angle H_{zT}(e^{j\omega}) = \angle\vec{a} + \angle\vec{b} - (\angle\vec{c} + \angle\vec{d})$$

let $\vec{a} = e^{j\omega}$, $\vec{b} = e^{j\omega} - 1$

$\vec{c} = (e^{j\omega} - j/\sqrt{2})$, $\vec{d} = (e^{j\omega} + j/\sqrt{2})$

then $|H_{zT}(e^{j\omega})| = \dfrac{|\vec{a}|\,|\vec{b}|}{|\vec{c}|\,|\vec{d}|}$

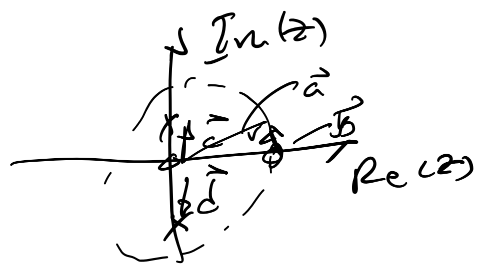
What does a single root contribute to the magnitude of $H_{zT}(e^{j\omega})$?



root factor: $e^{j\omega} - \vec{p}$

what is $|e^{j\omega} - \vec{p}|$?

recall $H_{ZT}(e^{j\omega}) = \dfrac{\vec{a}\,\vec{b}}{\vec{c}\,\vec{d}}$



$$|H_{ZT}(e^{j\omega})| = \frac{|\vec{a}||\vec{b}|}{|\vec{c}||\vec{d}|} = \frac{1\,|e^{j\omega}-1|}{|e^{j\omega}-\frac{j}{\sqrt{2}}||e^{j\omega}+\frac{j}{\sqrt{2}}|}$$