ECE 438 Lecture 6 February 2023

Announcements

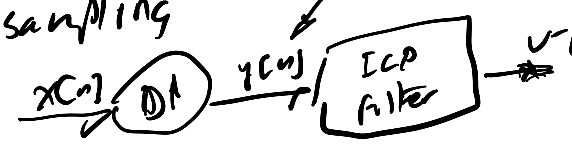
- Office Hour today at 4p EST via Zoom
- Quiz No. 1 will be released today at 6p EST
  You will have 30 minutes to complete it. It must be submitted by 11:59p EST today.

Sampling Rate Conversion in DT

Module 1.4.3

$Y(\omega) = X(D\omega)$

Upsampling



Interpolator

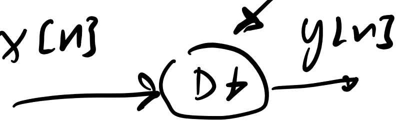
In time domain, we have a DT version of WKS sampling expansion

stretch signal in time domain, but do not lose any information

∴ in frequency domain, have no aliasing and simply compress $X(\omega)$

Down sampling

$y[n] = x[Dn]$



D - integer

Example: $D=2$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[Dn] e^{-j\omega n}$$

What are we looking for?

want to see something that looks like

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n}$$

let $m = Dn \Rightarrow n = \frac{m}{D}$

$$Y(\omega) = \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega \frac{m}{D}}$$

($m = Dn$, where n is an integer)

to get rid of the restriction, we employ a trick

let $S_D[n] = \begin{cases} 1, & n = Dk \\ 0, & \text{else} \end{cases}$ — integer k

$$Y(\omega) = \sum_{m=-\infty}^{\infty} S_D[m] x[m] e^{-j\omega \frac{m}{D}}$$

What is $S_D[m]$?

$$S_D[m] = \frac{1}{D} \sum^{D-1} e^{j 2\pi \frac{m}{D} k}$$ — note $S_D[m]$ repeats with periodicity

$D \quad k=0$

Verify that this works

$$= \frac{1}{D} \frac{1-e^{+j2\pi m}}{1-e^{+j2\pi \frac{m}{D}}}$$ Geometric series

$$1-e^{j2\pi m k} = 0$$

So $S_D[m] = 0$ unless denominator is zero, which is true whenever m is a multiple of D

So we just need to know $\lim_{m \to 0} S_D[m]$

Expand both numerator and denominator in a Taylor series

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\lim_{m \to 0} S_D[m] = \frac{1}{D} \frac{j2\pi m}{j2\pi m/D} = \frac{D}{D} = 1$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} S_D[n]\, x[n]\, e^{-j\omega n/D}$$

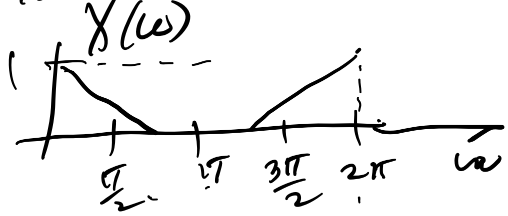
$$= \sum_{n=-\infty}^{\infty} \frac{1}{D} \sum_{k=0}^{D-1} e^{+j2\pi \frac{kn}{D}} x[n]\, e^{-j\omega n/D}$$

Switch the summations !!

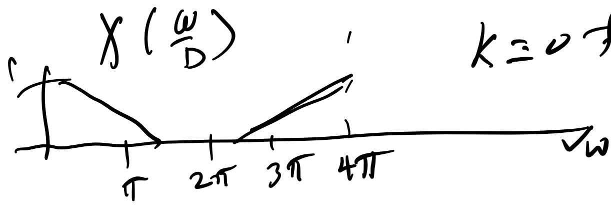
$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} \left\{ \sum_{m=-\infty}^{\infty} \frac{x[m] e^{+j\frac{2\pi k m}{D}} e^{-j\omega m/D}}{e^{-jm\left(\frac{\omega}{D} - \frac{2\pi k}{D}\right)}} \right\}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega - 2\pi k}{D}\right)$$

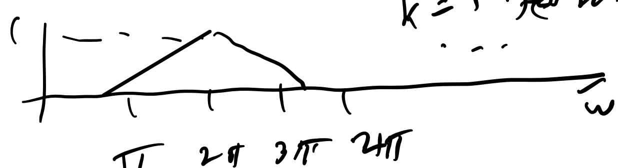
What does this look like?



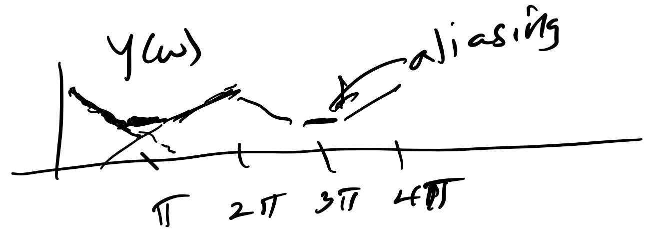
$D = 2$



k = 0 term in the summation



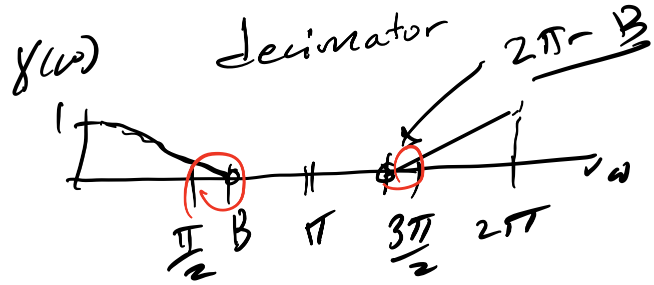
add them to get $Y(\omega)$ & multiply by $\frac{1}{D}$



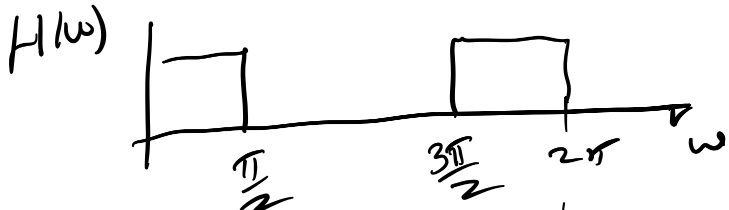
To prevent aliasing, we need to prebandlimit $X(\omega)$ to maximum frequency of red. sample with a digital filter before we downsample

$$x[n] \rightarrow \boxed{H_{LP}(\omega)} \xrightarrow{v[n]} \boxed{D\downarrow} \rightarrow y[n]$$

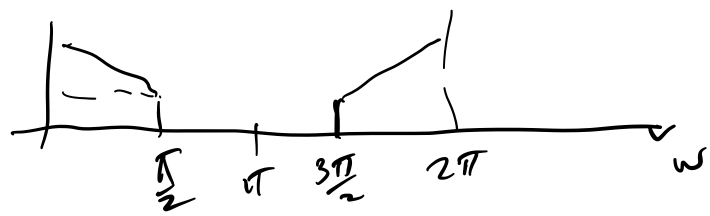
decimator



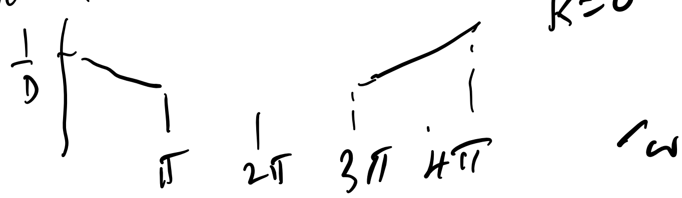
$B > \frac{\pi}{2}$



$V(\omega) = H(\omega) X(\omega)$

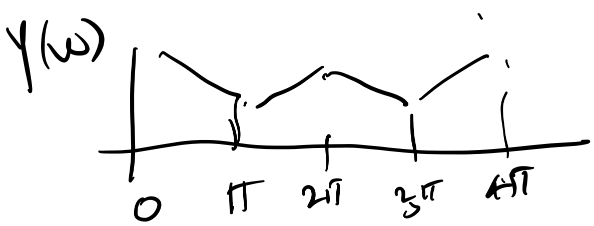


$Y(\omega)$ ($k = 0$ term only)



$k=0$

$D = 2$



$k=1$

New topic Z transform (ZT) Module 1.5

2-sided ZT ⟸ new??

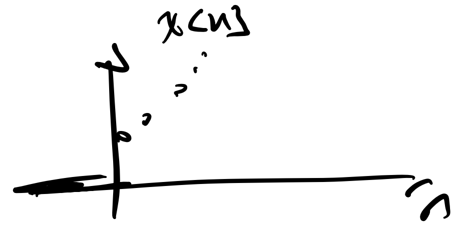
definition $X_{ZT}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= X_{DTFT}(\omega)\Big|_{z = e^{+j\omega}}$$

$$= X_{ZT}(e^{j\omega})$$ ⇐ This is how it is written in the legacy notes

Example 1

$$x[n] = 2^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N-1} \left(\frac{2}{z}\right)^n = \lim_{N \to \infty} \frac{1 - \left(\frac{2}{z}\right)^N}{1 - \left(\frac{2}{z}\right)}$$

$$= \frac{1}{1 - \frac{2}{z}}, \quad |z| > 2$$

$$= \frac{1}{1 - 2z^{-1}}, \quad |z| > 2$$

Region of convergence (ROC) for this Z-transform is $|z| > 2$