ECE 438 Lecture 24 February 2023

Announcements:

- See e-mails with heading: ECE 438 update with information about recently posted materials

Continue with FFT algorithm

Module 1.6.4.1 — FFT development

1.6.4.2 — History of FFT

Seeking an efficient way to evaluate the DFT

$$X^{(N)}[k] = \sum_{n=0}^{N-1} x^{(N)}[n] e^{-j2\pi kn/N} \qquad (1)$$

def 1 complex operation (c.o.) as one complex addition + one complex multiplication

Direct evaluation the above sum requires

$$C_{DIRECT} \cong N^2 \text{ c.o.}$$

Assume N is even

define $x_0^{(N/2)}[m] = x^{(N)}[2m]$, $m=0,\dots,\frac{N}{2}-1$

$x_1^{(N/2)}[m] = x^{(N)}[2m+1]$, $m=0,\dots,\frac{N}{2}-1$

Can write (1) as

$$X^{(N)}[k] = \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x_0^{(N/2)}[m] e^{-j2\pi \frac{mk}{N/2}}}_{} +$$

$$e^{-j2\pi \frac{k}{N}} \sum_{m=0}^{\frac{N}{2}-1} x_1^{(N/2)}[m] e^{-j2\pi \frac{mk}{N/2}}$$

$e^{-j\frac{2\pi k}{N}} \sum_{n=0} x_1$ large

twiddle factor

let $x_0^{(N/2)}[n] \xrightarrow{N/2\text{-DFT}} X_0^{(N/2)}[k]$, $k=0,\dots,\frac{N}{2}-1$

$x_1^{(N/2)}[n] \xrightarrow{\frac{N}{2}\text{ pt. DFT}} X_1^{(N/2)}[k]$, $k=0,\dots,\frac{N}{2}-1$

then $X^{(N)}[k] = X_0^{(N/2)}[k] + e^{-j\frac{2\pi k}{N}} X_1^{(N/2)}[k]$

$k=0,\dots,N-1$

Consider computation

Doing $2\ \frac{N}{2}$ pt. DFT $C_{DIRECT}^{(N/2)} \cong \left(\frac{N}{2}\right)^2 2 + N$

$\approx \frac{N^2}{4}$ for large N

Now we have $\frac{N^2}{2}$ c.o. instead of N^2 c.o.'s

Can we do better?

Yes - assume $\frac{N}{2}$ is even!

$\Rightarrow$ we have two $\frac{N}{2}$ DFTs each requiring

$\frac{N^2}{4}$ computations

Should get an additional factor of 2 savings

so we should have

$\frac{N^2}{4}$ c.o. for N-pt. DFT

Consider limiting case $\Rightarrow N=2^m$

Specifically, consider $N=2^3=8$

$X^{(8)}[k] = X_0^{(4)}[k] + e^{-j\frac{2\pi k}{8}} X_1^{(4)}[k]$

$$x_0^{(4)}[m] = x^{(8)}[2m], \quad m = 0, \dots, 3$$

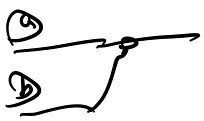
$$x_1^{(4)}[m] = x^{(8)}[2m+1], \quad m = 0, \dots, 3$$

the $x_0^{(4)}[m] \xrightarrow{\text{4 pt. DFT}} X_0^{(4)}[k]$

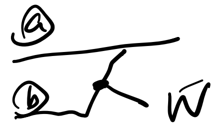
$x_1^{(4)}[m] \xrightarrow{\text{4 pt. DFT}} X_1^{(4)}[k]$

put this in flow diagram

conventions:

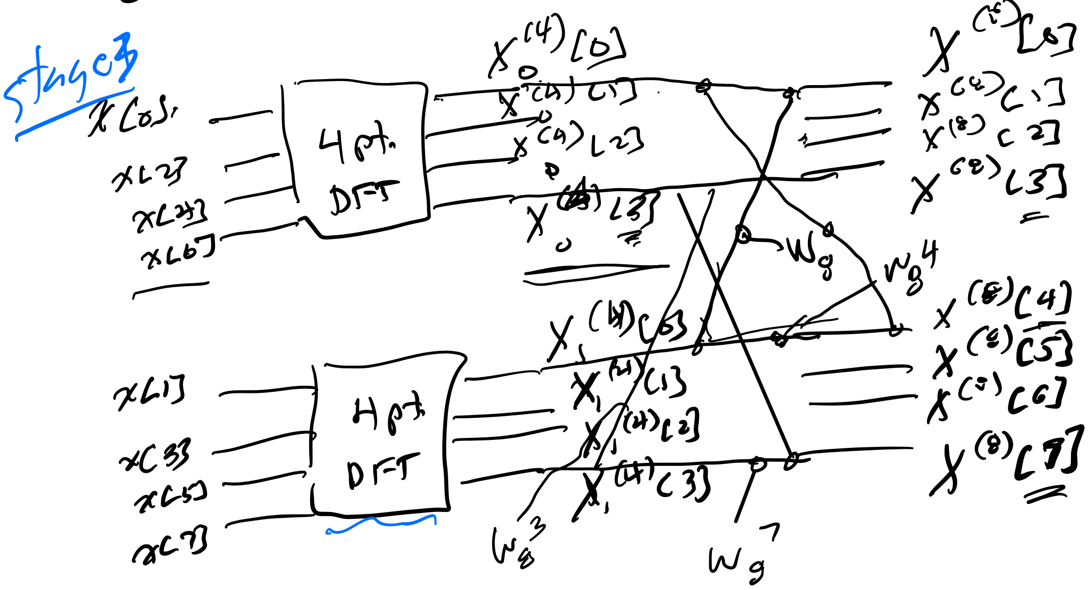
1) 

sum a + b

2) 

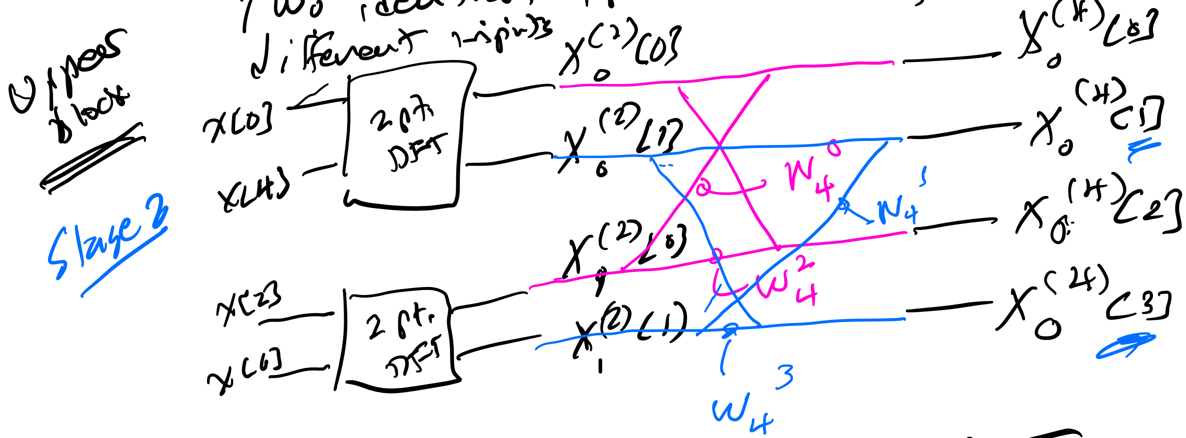
multiply bottom line by W

define $W_N = e^{j\frac{2\pi}{N}}$



What does 4 pt. DFT look like?

Two identical 4 pt. DFT blocks, but with different inputs



Consider 2 pt. DFTs:

Stage 1

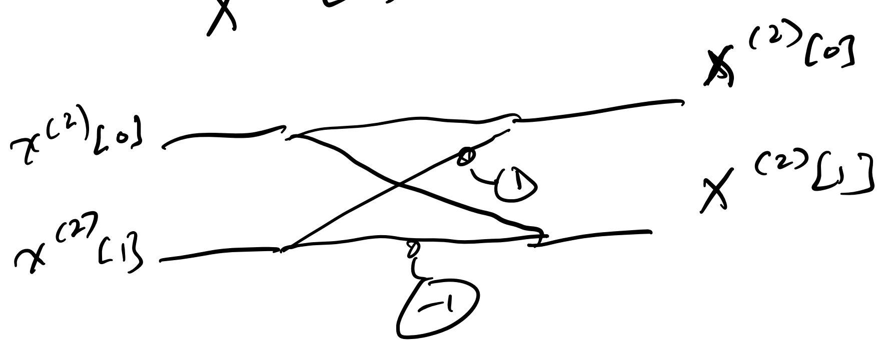
$$X^{(2)}[k] = \sum_{n=0}^{1} x^{(2)}[n] e^{-j2\pi nk/2}$$

$$e^{-j\pi nk} = (1 \text{ or } -1)$$

So

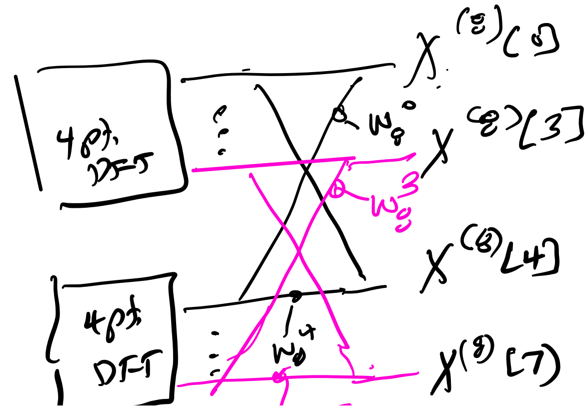
$$X^{(2)}[0] = x^{(2)}[0] + x^{(2)}[1]$$

$$X^{(2)}[1] = x^{(2)}[0] - x^{(2)}[1]$$

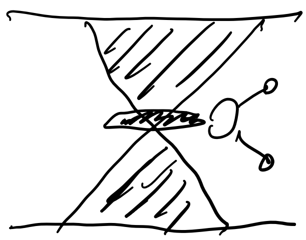


Summarize

Stage 3



W_y



butterfly

Each stage consists of $\frac{N}{2}$ butterflies

Each butterfly requires 2 complex additions + 2 complex multiplications $\Rightarrow$ 2 C.O.

$\Rightarrow$ Each stage requires N C.O.

In general, there are $\log_2(N)$ stages where $N = 2^m$

So total computation for full FFT is

$$C_{FFT} = N \log_2(N)$$

Consider some example values for N:

N	C_{DIRECT}	$\log_2(N)$	C_{FFT}	η (gain) $= \frac{C_{DIRECT}}{C_{FFT}}$
2	4	1	2	2
16	256	4	64	4

eta

$C_{DIRECT} = N^2 \quad C_{FFT} = N \log_2(N)$