ECE 438 lecture 22 February 2023

- Office Hour today 3:30p EST
- HW #4 due today on Gradescope at 11:59p EST

DFT $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}, \quad k=0,\dots,N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, \quad n=0,\dots,N-1$$

Application to spectral analysis

Module 1.6.3

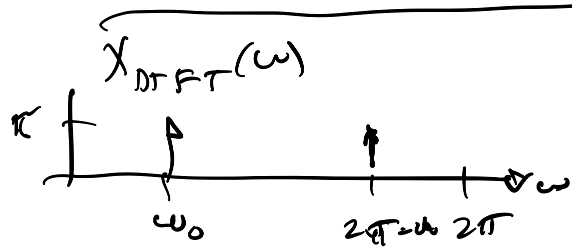
1. need to truncate signal to $n=0,\dots,N-1$
   $\Rightarrow$ leakage
2. need to sample frequency $\omega_k = \frac{2\pi k}{N}, k=0,\dots,N-1$
   $\Rightarrow$ picket fence

start with DTFT

$$x[n] = \cos(\omega_0 n)$$

$$X_{DTFT}(\omega) = \pi\{\delta(\omega-\omega_0) + \delta(\omega-(2\pi-\omega_0))\}$$

$$0 \le \omega \le 2\pi$$



Truncate $x[n]$ to include only samples for $n=0,\dots,N-1$

$$x_{tr}[n] = \begin{cases} x[n], & n=0,\dots,N-1 \\ 0, & \text{else} \end{cases}$$

define $W[n] = \begin{cases} 1, & n=0,\dots,N-1 \\ 0, & \text{else} \end{cases}$

letter W not ω

have $x_{tr}[n] = w[n]\,x[n], \quad \forall n$

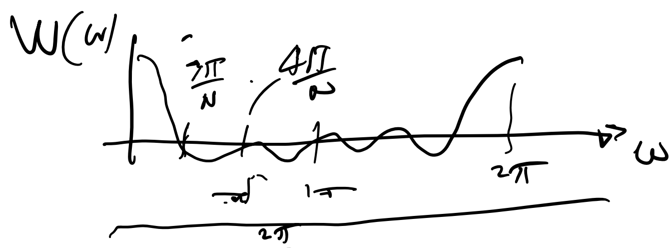
$$X^{tr}_{DTFT}(\omega) = \frac{1}{2\pi}\int_0^{2\pi} W(\mu)\, X_{DFT}(\omega-\mu)\, d\mu$$

need to know $W(\mu)$

have shown previously that

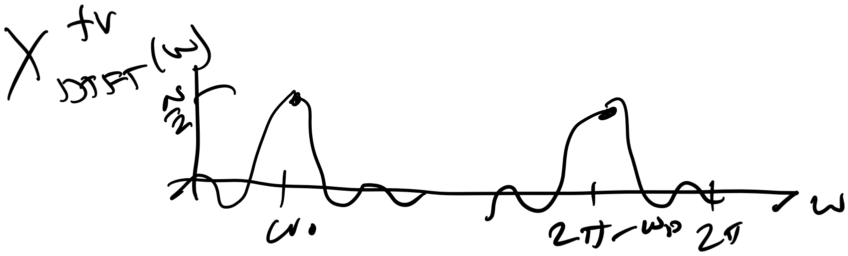
$$W(\omega) = e^{-j2\pi\omega\frac{(N-1)}{2}}\, psinc_N(\omega)$$

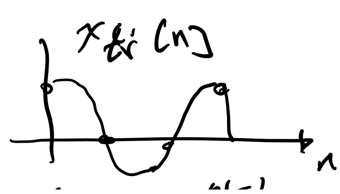
Ignoring the phase factor have:



$$X^{tr}_{DTFT}(\omega) = \frac{1}{2\pi}\int_0^{2\pi} W(\mu)\left[\pi\left(\delta(\omega-\mu-\omega_0) + \delta(\omega-\mu-(2\pi-\omega_0)\right)\right] d\mu$$

$$= \frac{1}{2}\left\{ W(\omega-\omega_0) + W(\omega-(2\pi-\omega_0)) \right\}$$





0 $N-1$

Module 1.6.3

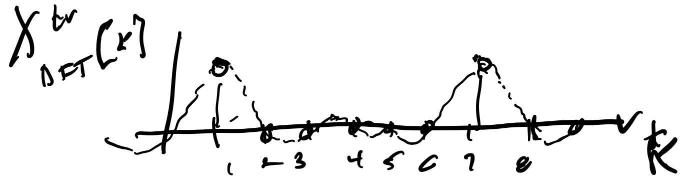
Know that $X_{DFT}^{tr}[k] = X_{DTFT}^{tr}\left(\frac{2\pi k}{N}\right)$

i.e. sample in frequency domain

Examples

① $\omega_0 = \frac{2\pi k_0}{N}$ for integer k_0 ; i.e. ω_0 coincides with a frequency sampling point

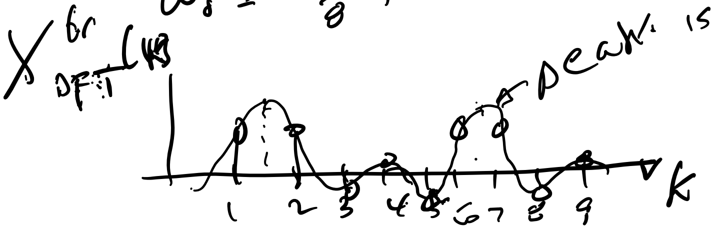
let $k_0 = 1$, $N = 8$ $\Rightarrow$ $\omega_0 = \pi/4$



this is exactly what we want!

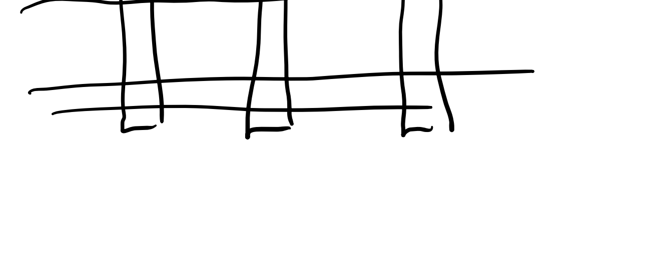
② $\omega_0 = \frac{2\pi k_0}{N} + \frac{\pi}{N}$, $N = 8$

$\omega_0 = \frac{3\pi}{8}$, $k_0 = 1$



leakage

this effect is called picket fence



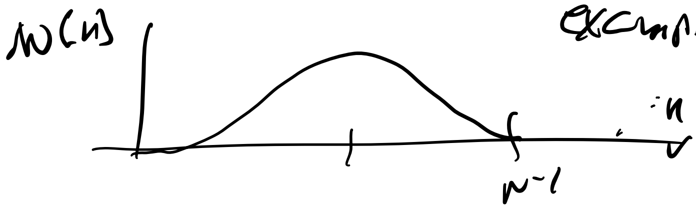
picket fence

See legacy notes for $x_{tr}[n]$ in this case (example 2)

What happens as N increases?

Gibbs phenomena tells us that increasing N will decrease width of sidelobes ($\frac{2\pi}{N}$), but not their amplitude

To decrease amplitude of the sidelobes need to use a smoothly tapering window like:



example = raised cosine

tradeoff: as I decrease amplitude of sidelobes, I increase their width

For optimum window, see Kaiser window

related to windowing technique for FIR filter design

New topic: Fast Fourier transform (FFT) algorithm

~~Module~~ 1.6.4.1 - Derivation of FFT

1.6.4.2 - History of FFT

Review DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad k = 0, \ldots, N-1$$

How much computation?

① For each value of k, have to perform N complex multiplications (ignore fact the $e^{-j2\pi kn/N}$ might equal $-1, 1, -j$, or j)

(Signal $x[n]$ may also be complex

② Sum N complex-valued terms $\Rightarrow N-1$ complex additions. Assume N is large $\Rightarrow N-1 \approx N$

define complex-operation as 1 complex multiplication + 1 complex addition (c.o.)

Summarizing, for each k have N c.o.s

Repeat this for $k = 0, \ldots, N-1 \Rightarrow$

$$C_{DIRECT} = N^2 \text{ c.o.s}$$

let's assume N is even

$$X^{(N)}[k] = \sum_{\substack{n=0 \\ n\text{-even}}}^{N-1} x[n] e^{-j2\pi \frac{nk}{N}} + \sum_{\substack{n=0 \\ n\text{-odd}}}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$$

let $n = 2m$

$m = 0, \ldots, \frac{N}{2} - 1$

when $m = \frac{N}{2} - 1$, $n = N - 2$

let $n = 2m + 1$

$m = 0, \ldots, \frac{N}{2} - 1$

when $m = 0$, $n = 1$

when $m = \frac{N}{2} - 1$, $n = N - 1$

define

$x_0[m] = x[2m]$, $m = 0, \ldots, \frac{N}{2} - 1$

$x_1[m] = x[2m+1]$, $m = 0, \ldots, \frac{N}{2} - 1$

$$X^{(N)}[k] = \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x_0[m] e^{-j2\pi \frac{mk}{(N/2)}}}_{\frac{N}{2} \text{ pt. DFT of } x_0[m]} + e^{-j\frac{2\pi k}{N}} \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x_1[m] e^{-j2\pi \frac{mk}{(N/2)}}}_{\frac{N}{2} \text{ pt. DFT of } x_1[m]}$$